Computational techniques for discrete-time renewal processes

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ABSTRACT: For applying life-cycle costing, the total costs should be computed over a bounded or unbounded time horizon. In order to determine the expected costs of maintenance, renewal theory can be applied when we can identify renewals that bring a component back into the as-good-as-new condition. This paper presents useful computational techniques to determine the probabilistic characteristics of a renewal process. Because continuous-time renewal processes can be approximated with discrete-time renewal processes, it focusses on the latter processes. It includes methods to compute the probability distribution, expected value and variance of the number of renewals over a bounded time horizon, the asymptotic expansions for the expected value and variance of the number of renewals over an unbounded time horizon, the approximation of a continuous renewal-time distribution with a discrete renewal-time distribution, and the extension of the discrete-time renewal model with the possibility of zero renewal times (in order to cope with an upper-bound approximation of a continuous-time renewal process).

1 INTRODUCTION

Life-cycle costing concerns the estimation of the total costs from the design and building, via the use, to the disposal of a component over a structure's or system's lifespan. For example, the design life of a bridge is often in the order of eighty to one hundred years. For applying life-cycle costing, the total cost should generally be computed over a bounded or unbounded time horizon. For this purpose, several cost-based criteria can be used such as the expected life-cycle cost averaged or discounted over a bounded or unbounded time horizon (van Noortwijk, 2003). In order to determine the expected costs of maintenance actions, renewal theory has proven its usefulness. Renewal theory can be applied when we can identify renewals that bring a component back into the as-good-as-new condition. Usually, the total life-cycle cost must be estimated while taking account of the large uncertainties in the renewal times. We focus on fixed renewal cost not depending on the renewal time. In order to compute the probability distribution of the costs over a bounded or unbounded time horizon, it is therefore sufficient to compute the probability distribution of the number of renewals and multiply them with the renewal cost.

This paper presents useful computational techniques to determine the probabilistic characteristics of a renewal process. Because continuous-time renewal processes can be approximated with discrete-time renewal processes, it focusses on the latter processes. It includes methods to compute the probability distribution, expected value and variance of the number of renewals in a bounded time horizon, the asymptotic expansions for the expected value and variance of the number of renewals over an unbounded time horizon, the approximation of a continuous renewal-time distribution with a discrete renewal-time distribution, and the extension of the discrete-time renewal model with the possibility of zero renewal times. For computing the probability distribution of the number of renewals, we compare the recursive convolution summation with the recursion formula of De Pril (1985). The asymptotic expansions for the first and second moment of the number of renewals for discrete-time and continuous-time renewal processes are derived by Feller (1949) and Smith (1954, 1958), respectively.

This paper is organised as follows. In Section 2, we present techniques to compute probabilistic characteristics of the number of renewals. These computational techniques are applied in Section 3. Conclusions are given in Section 4.

2 DISCRETE-TIME RENEWAL PROCESSES

This section presents techniques to compute probabilistic characteristics of the number of renewals.

2.1 Expected value of number of renewals

Let us consider a discrete renewal process, whereby renewals bring the system back into its original condition or as-good-as-new state. After each renewal we start (in a statistical sense) all over again. A discrete-time renewal process $\{N(t), t = 1, 2, \ldots\}$ is a non-negative integer-valued stochastic process that registers the successive renewals in the time interval $(0, t]$. Let the renewal inter-occurrence times $T_1, T_2, \ldots,$ be positive, independent, identically distributed, random quantities having the discrete probability function

$$\mathbb{P}(T_k = i) = p_i, \;\; i = 1, 2, \ldots, \;\; \sum_{i=1}^{\infty} p_i = 1,$$

where p_i represents the probability of a renewal in unit time i. Denote the renewal occurrence times as $S_n = \sum_{k=1}^{n} T_k, n = 1, 2, \ldots$ Notice that, because $T_k \geq 1$ for all k, the partial sums S_n obey the inequality $S_n \geq n$. In terms of the occurrence times, the renewal process $\{N(t), t = 1, 2, \ldots\}$ can be defined as:

$$N(t) = \max\{j | S_j \leq t\} = \sum_{k=1}^{\infty} 1_{\{S_k \leq t\}}, \qquad (1)$$

$t = 1, 2, \ldots$, where 1_A denotes the indicator function of the set A. The expected number of renewals in the bounded time horizon $(0, t]$, denoted by $\mathbb{E}(N(t))$, can then be written as the recursive equation

$$\mathbb{E}(N(t)) = \sum_{i=1}^{t} \left[1 + \mathbb{E}(N(t-i))\right] p_i \qquad (2)$$

for $t = 1, 2, 3\ldots$ and $N(0) \equiv 0$. To obtain this equation, we condition on the values of the first renewal time T_1 having the value i and apply the law of total probability:

$$\mathbb{E}(N(t)) = \sum_{i=1}^{t} \mathbb{E}\left(1 + \sum_{k=2}^{\infty} 1_{\{i+T_2+\ldots+T_k \leq t\}}\right) p_i$$

for $t = 1, 2, 3\ldots$

2.2 Probability distribution of number of renewals

In addition to the expected number of renewals, it is useful to compute the probability distribution of the number of renewals in time interval $(0, t]$. For this purpose, we compare the traditional recursive convolution summation with the recursion formula developed by De Pril (1985). This formula is known in the field of insurance risk and ruin (Dickson, 2005).

Recall that the renewal inter-occurrence time T_k is defined on the positive integers and denote its probability function as

$$\mathbb{P}(T_k = i) = p_i = p_T(i), \quad i = 1, 2, \ldots$$

The probability distribution of S_n is given by the n-fold convolution

$$\mathbb{P}(S_n = s) = p_{S_n}(s) = p_T^{n*}(s).$$

The usual way to compute the n-fold convolution of an arithmetic distribution with itself at s, $p_T^{n*}(s)$, is by successive application of the formula

$$p_T^{(k+1)*}(r) = \sum_{i=1}^{r} p_T(i) p_T^{k*}(r-i), \quad p_T(0) = 0, \qquad (3)$$

for $k = 1, 2, \ldots, n-1$, and $r = 1, 2, \ldots, s$.

An alternative way to compute the convolution of an arithmetic distribution is the recursion formula of De Pril (1985). Let $p_T(1) > 0$, then, according to De Pril (1985), the following recursion holds:

$$p_{S_n}(n) \;=\; [p_T(1)]^n, \qquad (4)$$

$$p_{S_n}(s) \;=\; \sum_{x=1}^{s-n} \left(\frac{n+1}{s-n} x - 1\right) \frac{p_T(x+1) p_{S_n}(s-x)}{p_T(1)},$$

for $n = 1, 2, \ldots$ and $s = n+1, n+2, \ldots$ Hence,

$$p_{S_n}(s) = 0$$

for $n = 1, 2, \ldots$ and $s = 1, \ldots, n-1$.

Using the equivalence of the two events

$$N(t) = n \Leftrightarrow S_n \leq t < S_{n+1}, \qquad (5)$$

it follows that the probability distribution of $N(t)$ can be written as

$$\mathbb{P}(N(t) = n) = \mathbb{P}(S_n \leq t) - \mathbb{P}(S_{n+1} \leq t) \qquad (6)$$

for $t, n = 1, 2, \ldots$ and

$$\mathbb{P}(N(t) = 0) = 1 - \mathbb{P}(S_1 \leq t) = \mathbb{P}(T_1 > t) \qquad (7)$$

for $t = 1, 2, \ldots$ Note that for fixed n, where $n = 1, 2, \ldots$, we have

$$\mathbb{P}(S_n = t) = p_{S_n}(t) = 0$$

for $t = 1, \ldots, n-1$. Hence,

$$\mathbb{P}(N(t) = n) = \mathbb{P}(S_n \leq t) - \mathbb{P}(S_{n+1} \leq t) = 0$$

for $t = 1, \ldots, n-1$.

De Pril (1985) generalised Equation (4) to integer-valued T and the condition $p_T(1) > 0$ replaced by

$$p_T(m) > 0, \quad m = \min\{i : p_T(i) > 0\}.$$

Under this condition, the revised recursion reads:

$$p_{S_n}(mn) = [p_T(m)]^n,$$

and

$$p_{S_n}(s) = \tag{8}$$
$$= \sum_{x=1}^{s-mn} \left(\frac{n+1}{s-mn}x - 1\right) \frac{p_T(x+m)p_{S_n}(s-x)}{p_T(m)},$$

for $n = 1, 2, \ldots$ and $s = mn+1, mn+2, \ldots$

With respect to the number of algebraic operations, De Pril's method is preferred over the traditional convolution if we need to determine the probability distribution of S_n for one particular n (Sundt and Dickson, 2000). If we need to determine the probability distributions of $S_1, \ldots, S_{n-1}$ as well, then the traditional evaluation is preferred.

2.3 *Number of renewals in a unit of time*

Using the expected number of renewals in a bounded time horizon $(0, t]$ in Equation (2), we can easily determine the expected number of renewals in unit of time t:

$$\mathbb{E}(N(t) - N(t-1)) = \mathbb{E}(N(t)) - \mathbb{E}(N(t-1)) \tag{9}$$

for $t = 1, 2, \ldots$ This equation is not only useful for determining the expected value of the number of renewals in a particular unit of time, but it can also be used to determine the probability distribution of the number of renewals in this unit of time in a very straightforward manner. Due to the definition of the discrete-time renewal process, there can be either no or one renewal per unit time (recall that for every inter-occurrence renewal time T, the inequality $T \geq 1$ holds). This implies that

$$\mathbb{P}(N(t) - N(t-1) = n) = 0$$

for $t = 1, 2, \ldots$ and $n = 2, 3, \ldots$ The expected value of the number of renewals in unit time t can then be written as

$$\mathbb{E}(N(t) - N(t-1)) =$$
$$= \sum_{n=0}^{\infty} n\mathbb{P}(N(t) - N(t-1) = n) \tag{10}$$
$$= \mathbb{P}(N(t) - N(t-1) = 1)$$

for $t = 1, 2, \ldots$ Hence, for the probability distribution of $N(t) - N(t-1)$, we can summarise:

$$\mathbb{P}(N(t) - N(t-1) = n) =$$
$$= \begin{cases} 1 - \mathbb{E}(N(t) - N(t-1)), & n = 0, \\ \mathbb{E}(N(t) - N(t-1)), & n = 1, \\ 0, & n = 2, 3, \ldots \end{cases}$$

2.4 *Variance of number of renewals in time interval*

In order to compute the variance of the number of renewals in time interval $(0, t]$, the second moment of the number of renewals remains to be determined. The first and second moment of the number of renewals in time interval $(0, t]$, denoted by $\mathbb{E}(N(t))$ and $\mathbb{E}(N^2(t))$, respectively, solve Equation (2) as well as the recursive equation

$$\mathbb{E}(N^2(t)) = \sum_{i=1}^{t} p_i \mathbb{E}([1 + N(t-i)]^2) = \tag{11}$$
$$= \sum_{i=1}^{t} p_i \left[1 + 2\mathbb{E}(N(t-i)) + \mathbb{E}(N^2(t-i))\right]$$

for $t = 1, 2, 3, \ldots$ This equation is obtained by conditioning on the values of the first renewal time T_1 (van Noortwijk, 2003):

$$\mathbb{E}(N^2(t)) = \sum_{i=1}^{t} \mathbb{E}\left(\left[1 + \sum_{k=2}^{\infty} 1_{\{i+T_2+\ldots+T_k \leq t\}}\right]^2\right) p_i$$

for $t = 1, 2, 3 \ldots$ The variance of the number of renewals in time interval $(0, t]$ can now easily be derived by combining Equations (2) and (11); that is,

$$\text{Var}(N(t)) = \mathbb{E}(N^2(t)) - [\mathbb{E}(N(t))]^2$$

for $t = 1, 2, 3, \ldots$

2.5 *Asymptotic properties of number of renewals*

Using the discrete renewal theorem (Feller, 1950, Chapters 12 & 13; Karlin & Taylor, 1975, Chapter 3), the long-term expected average number of renewals per unit time is

$$\lim_{t \to \infty} \frac{\mathbb{E}(N(t))}{t} = \frac{1}{\sum_{i=1}^{\infty} ip_i} = \frac{1}{\mathbb{E}(T)}. \tag{12}$$

Let a renewal cycle be the time period between two renewals, then the denominator can be recognised as the expected cycle length $\mathbb{E}(T)$ (mean lifetime). The limit in Equation (12) exists provided that the greatest common divisor of the integers $i = 1, 2, 3, \ldots$ for which $p_i > 0$ is equal to unity. The simplest assumption assuring this is $p_1 > 0$.

Similarly, the long-term average variance of the number of renewals per unit time is:

$$\lim_{t \to \infty} \frac{\text{Var}(N(t))}{t} = \frac{\text{Var}(T)}{[\mathbb{E}(T)]^3}. \tag{13}$$

A proof of this theorem is given in Feller (1949). He also showed that, as $t \to \infty$, $N(t)$ is asymptotically normal with mean

$$\mathbb{E}(N(t)) \sim \frac{t}{\mathbb{E}(T)} \tag{14}$$

and variance

$$\mathrm{Var}(N(t)) \sim \frac{t\mathrm{Var}(T)}{[\mathbb{E}(T)]^3}. \quad (15)$$

For discrete-time renewal processes, the asymptotic expression for the mean of the number of renewals in Equation (14) follows from (see Feller, 1949)

$$\lim_{t\to\infty}\left\{\mathbb{E}(N(t)) - \frac{t}{\mathbb{E}(T)}\right\} = \quad (16)$$
$$= \frac{\mathbb{E}(T^2)}{2[\mathbb{E}(T)]^2} + \frac{1}{2\mathbb{E}(T)} - 1.$$

For T having a geometric distribution, the right-hand side of Equation (16) equals zero and the asymptotic expansion is exact (see also Section 3.1). The asymptotic expansion (16) can be used to approximate the mean of the number of renewals over a bounded time horizon.

For continuous-time renewal processes, the asymptotic expression is slightly different and was derived by Smith (1954):

$$\lim_{t\to\infty}\left\{\mathbb{E}(N(t)) - \frac{t}{\mathbb{E}(T)}\right\} = \frac{\mathbb{E}(T^2)}{2[\mathbb{E}(T)]^2} - 1. \quad (17)$$

Note that Feller's asymptotic expansion for the discrete-time case can be rewritten in terms of Smith's asymptotic expansion for the continuous-time case as follows:

$$\lim_{t\to\infty}\left\{\mathbb{E}(N(t)) - \frac{t+\frac{1}{2}}{\mathbb{E}(T)}\right\} = \frac{\mathbb{E}(T^2)}{2[\mathbb{E}(T)]^2} - 1.$$

2.6 Discretisation of renewal-time distribution

Although the above-mentioned computational methods are developed for discrete-time renewal processes, they also can be applied to continuous-time renewal processes if the probability distribution of the renewal time is properly discretised. For this purpose, Dickson (2005) presents three methods to discretise a continuous renewal-time distribution

$$\mathbb{P}(T \le t) = F(t), \ \ t \ge 0, \quad F(0) = 0.$$

First, discretisation of the continuous renewal-time distribution by matching probabilities:

$$p_i = F(i) - F(i-1), \quad i = 1, 2, \ldots \quad (18)$$

The advantage of this approximation is that for the integers $x = 0, 1, 2, \ldots$, the probability distributions are equal; that is,

$$P(x) = \sum_{i=1}^{x} p_i = F(x).$$

Note that for non-integer values of $x > 0$, $P(x) < F(x)$. Therefore, this approximation represents a lower-bound approximation of the continuous renewal-time distribution.

Second, discretisation of the continuous renewal-time distribution by creating the following upper-bound approximation:

$$\tilde{p}_i = F(i+1) - F(i), \ \ i = 1, 2, \ldots, \ \ \tilde{p}_0 = F(1). \quad (19)$$

Hence,

$$\tilde{P}(x) = \sum_{i=0}^{x} \tilde{p}_i = F(x+1) > F(x)$$

for $x = 0, 1, 2, \ldots$

Third, discretisation of the continuous renewal-time distribution by matching the mean:

$$\hat{P}(x) = \sum_{i=0}^{x} \hat{p}_i = \int_{x}^{x+1} F(y)\, dy.$$

for $x = 0, 1, 2, \ldots$ The approximate probability function $\hat{p}_i$, $i = 0, 1, 2, \ldots$, is mean preserving:

$$\sum_{i=0}^{\infty}[1 - \hat{P}(i)] = \sum_{i=0}^{\infty} \int_{i}^{i+1} (1 - F(y))\, dy = \mathbb{E}(T).$$

In the last two approximations, the continuous renewal-time distribution is discretised with respect to the non-negative integers $0, 1, 2, \ldots$ rather than $1, 2, \ldots$ Furthermore, it should be noted that the above discretisation method is not restricted to a unit-time length of one. To assure an accurate discretisation, the unit time should be chosen such that the probability of more than one renewal per unit time would be almost zero for the original continuous-time renewal process as well.

2.7 Renewals with possible zero renewal time

When a continuous renewal-time distribution is discretised on $0, 1, 2, \ldots$, the recursive renewal equation (2) must be reformulated with the possibility of

$$\mathbb{P}(T = 0) = p_0 > 0.$$

Hence, we extend the renewal process with the possible occurrence of a renewal at time zero; that is, let the renewal times $T_1, T_2, T_3, \ldots,$ be non-negative, independent, identically distributed, random quantities having the discrete probability function

$$\mathbb{P}\{T_k = i\} = p_i, \ \ i = 0, 1, 2, \ldots, \ \ \sum_{i=0}^{\infty} p_i = 1.$$

The corresponding renewal process is then $\{N(t), t = 0, 1, 2, \ldots\}$ with expected value satisfying the recursive equation

$$\mathbb{E}(N(t)) = \sum_{i=0}^{t} p_i \left[1 + \mathbb{E}(N(t-i))\right] \quad (20)$$

for $t = 1, 2, 3, \ldots$ and, using Equation (1),

$$\mathbb{E}(N(0)) = \sum_{k=1}^{\infty} \mathbb{P}\left(S_k = 0\right) = \sum_{k=1}^{\infty} p_0^k = \frac{p_0}{1-p_0}.$$

This renewal equation can be solved for the expected value $\mathbb{E}(N(t))$ by rewriting Equation (20) as follows

$$\mathbb{E}(N(t)) = \quad (21)$$
$$= \frac{p_0 + \sum_{i=1}^{t-1} p_i \left[1 + \mathbb{E}(N(t-i))\right]}{1-p_0} + \frac{p_t \cdot 1_{\{t \geq 1\}}}{(1-p_0)^2},$$

$t = 0, 1, 2, \ldots$ Clearly, Equation (2) follows when $p_0 = 0$. A similar equation can be derived for the second moment of the number of renewals over a bounded time horizon:

$$\mathbb{E}(N(t)^2) = \quad (22)$$
$$= \frac{p_0\left[1 + 2\mathbb{E}(N(t))\right]}{1-p_0} + \frac{p_t(1+p_0)}{(1-p_0)^3} \cdot 1_{\{t \geq 1\}} + \frac{\sum_{i=1}^{t-1} p_i \left[1 + 2\mathbb{E}(N(t-i)) + \mathbb{E}(N^2(t-i))\right]}{1-p_0},$$

$t = 0, 1, 2, \ldots$, where we used

$$\mathbb{E}(N^2(0)) = \sum_{k=0}^{\infty} k^2 p_0^k (1-p_0) = \frac{p_0(1+p_0)}{(1-p_0)^2}.$$

Equation (11) follows from (22) when $p_0 = 0$.

3 APPLICATIONS

This section presents examples in which probability distributions and expected values of the total life-cycle cost over a bounded time horizon are derived for renewal inter-occurrence times having a shifted geometric distribution, a discretised lifetime distribution based on a non-stationary gamma process, and a discretised lifetime distribution based on a stationary gamma process (resulting in a shifted Poisson distribution).

3.1 Shifted geometric distribution

Let the discrete inter-occurrence time of a natural hazard (like a flood or hurricane), denoted by T, be distributed as a shifted geometric distribution with parameter p, so that

$$\mathbb{P}(T = i) = p_i = (1-p)^{i-1} p, \;\; i = 1, 2, 3, \ldots \quad (23)$$

The parameter p can be interpreted as the probability of occurrence of a natural hazard per unit time.

For renewal inter-occurrence times having a shifted geometric distribution, the probability distribution of the number of renewals in a bounded time horizon t can be obtained in explicit form. It is a binomial distribution with parameters p and t:

$$\mathbb{P}(N(t) = n) = \binom{t}{n} p^n (1-p)^{t-n},$$

for $n = 0, 1, 2, \ldots, t$, and $t = 1, 2, 3, \ldots$ The mean and variance of the number of renewals in a bounded time horizon t are,

$$\mathbb{E}(N(t)) = tp$$

and

$$\text{Var}(N(t)) = tp(1-p),$$

respectively. Hence, the expected value and the variance of the number of natural hazards averaged over an unbounded time horizon are

$$\lim_{t\to\infty} \frac{\mathbb{E}(N(t))}{t} = p$$

and

$$\lim_{t\to\infty} \frac{\text{Var}(N(t))}{t} = p(1-p),$$

respectively. Using the first two moments of the shifted geometric distribution,

$$\mathbb{E}(T) = \frac{1}{p}, \quad \mathbb{E}(T^2) = \frac{2-p}{p^2},$$

we can prove that the right-hand side of Equation (17) equals zero and the asymptotic expansions are exact.

3.2 Discretised non-stationary gamma process

In structural engineering, a failure may be defined as the event in which the deteriorating resistance (or strength) drops below the stress. Because deterioration is uncertain, it is regarded as a time-dependent stochastic process $\{X(t),\ t \geq 0\}$ where $X(t)$ is a random quantity for all $t \geq 0$. For the stochastic deterioration process to be monotone, we can consider it as a gamma process. A gamma process is a stochastic process with independent non-negative increments having a gamma distribution with identical scale parameter. Abdel-Hameed (1975) was the first to propose the gamma process as a proper model for deterioration occurring random in time. The gamma process is suitable to model gradual damage monotonically accumulating over time, such as wear, fatigue,

Table 1: Parameters of corrosion model.

Quantity	Description	Value
$\hat{u}$	MLE scale parameter	16.3749
$\hat{a}$	MLE constant of proportionality	0.0538
$\hat{b}$	MLE exponent of power law	2.3718
r_0	Initial resistance	100%
s	Failure level	97%

corrosion, crack growth, erosion, consumption, creep, swell, etc.

In mathematical terms, the gamma process is defined as follows. Recall that a random quantity X has a gamma distribution with shape parameter $v > 0$ and scale parameter $u > 0$ if its probability density function is given by:

$$\text{Ga}(x|\, v, u) = \frac{u^v}{\Gamma(v)}\, x^{v-1} \exp\{-ux\}\, 1_{(0,\infty)}(x),$$

where $1_A(x) = 1$ for $x \in A$ and $1_A(x) = 0$ for $x \notin A$. Furthermore, let $v(t)$ be a non-decreasing, right continuous, real-valued function for $t \geq 0$ with $v(0) \equiv 0$. The gamma process with shape function $v(t) > 0$ and scale parameter $u > 0$ is a continuous-time stochastic process $\{X(t), t \geq 0\}$ with the following properties:

1. $X(0) = 0$ with probability one;
2. $X(\tau) - X(t) \sim \text{Ga}(v(\tau) - v(t), u)$ for $\tau > t \geq 0$;
3. $X(t)$ has independent increments.

Let $X(t)$ denote the deterioration at time t, $t \geq 0$, and let the probability density function of $X(t)$, in conformity with the definition of the gamma process, be given by

$$f_{X(t)}(x) = \text{Ga}\,(x\,|\, v(t), u) \tag{24}$$

with

$$\mathbb{E}(X(t)) = \frac{v(t)}{u}, \quad \text{Var}(X(t)) = \frac{v(t)}{u^2}. \tag{25}$$

Failure occurs when its deteriorating resistance, denoted by $R(t) = r_0 - X(t)$, drops below the stress s. We assume both the initial resistance r_0 and the stress s to be deterministic. Let the time at which failure occurs be denoted by the lifetime T. Due to the gamma distributed deterioration, Equation (24), the lifetime distribution can then be written as:

$$F(t) = \mathbb{P}(T \leq t) = \mathbb{P}(X(t) \geq r_0 - s) = \tag{26}$$
$$= \int_{x=r_0-s}^{\infty} f_{X(t)}(x)\, dx = \frac{\Gamma(v(t), [r_0 - s]u)}{\Gamma(v(t))},$$

where $\Gamma(a, x) = \int_{t=x}^{\infty} t^{a-1} e^{-t}\, dt$ is the incomplete gamma function for $x \geq 0$ and $a > 0$.

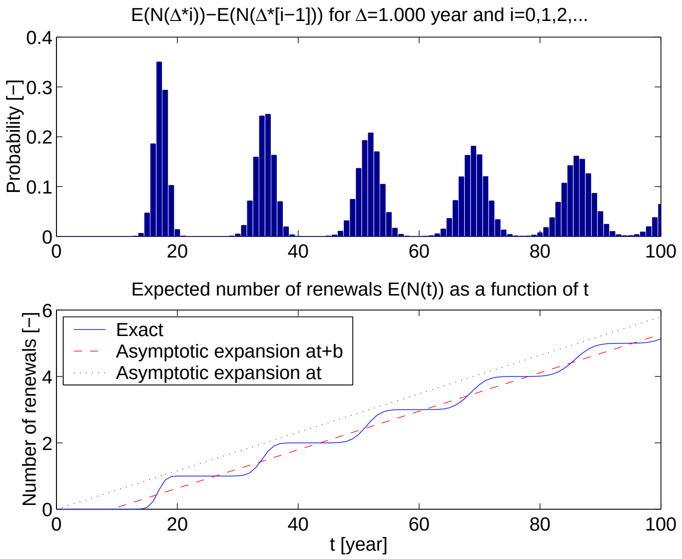

Figure 1: (Top) Probability of coating renewal per unit time of length $\Delta = 1$ year; (Bottom) Expected cumulative number of coating renewals.

Empirical studies show that the expected deterioration at time t is often proportional to a power law:

$$\mathbb{E}(X(t)) = \frac{v(t)}{u} = \frac{at^b}{u} \tag{27}$$

for some physical constants $u > 0$, $a > 0$, and $b > 0$. It should be noted that the gamma process is not restricted to using a power law for modelling the expected deterioration over time. As a matter of fact, any shape function $v(t)$ suffices, as long as it is non-decreasing, right continuous, and real-valued.

As an illustration, the corrosion process of the steel gates of the Dutch Haringvliet storm-surge barrier has been studied (taken from Heutink et al., 2004). To predict future corrosion, this process was fitted to five inspection data representing the percentage of the surface of a gate that has been corroded due to ageing of the coating. Every gate was inspected once at a different inspection interval. Assuming the five inspections to be independent, we determine the maximum-likelihood estimators (MLEs) of the parameters of the gamma process with expected deterioration being a power law in time; that is, we can determine $\hat{u}$, $\hat{a}$, and $\hat{b}$ (see Table 1 and Figure 7 of Frangopol et al. (2004)). The condition of a steel gate is defined as the percentage of non-corroded surface, where the initial condition is $r_0 = 100\%$ and the failure level is $s = 97\%$. The lifetime is defined as the time at which the cumulative amount of deterioration exceeds a corroded surface of $r_0 - s = 100 - 97 = 3\%$. As soon as the failure level is crossed, the coating is brought into a perfect condition again by thorough cleaning, grit blasting and airless spraying of the coating. This is done by spot repair as well as by totally removing and replacing the old coating system. If these operations are executed, serious environmental precautions are required.

The continuous lifetime distribution in Equation (26) is discretised with respect to units of time of length $\Delta = 1$ year by determining an upper bound of the cumulative distribution function (CDF) using Equation (19). For gamma-process stress-strength models, the advantage of applying the upper-bound (or lower-bound) CDF approximation is that the only information needed is the cumulative distribution function of the lifetime. As opposed to the probability density function and the moments, the CDF is available in explicit form. The convolution of the resulting discrete renewal-time distribution is determined by the recursive convolution summation in Equation (3) and De Pril's recursion formula in Equation (8). Unfortunately, because of the very small value of the first non-zero probability $p_6 = 3.3 \times 10^{-16}$, De Pril's recursion formula is unstable and doesn't give reasonable results (according to Equation (8), we have to divide by p_6). Therefore, we use the recursive convolution summation. The probability of a coating renewal per year is determined for every year in a design life of one hundred years and displayed in Figure 1 (top). For the bounded time horizon of one hundred years, the expected cumulative number of coating replacements, $\mathbb{E}(N(t))$, is determined using Equation (20) as well as approximated with the asymptotic expansions (14) and (16). In Figure 1 (bottom), we can clearly see that the asymptotic expansion including the constant term (16) is a much better approximation than the asymptotic expansion excluding the constant term (14). The asymptotic expansion including the constant term is especially useful when the renewal process have passed the transients and almost have settled down into an equilibrium.

3.3 *Discretised stationary gamma process*

As a simplified example, we study the replacement of a hydraulic cylinder on a swing bridge (adapted from van Noortwijk, 1998). Replacement brings the cylinder back into its as-good-as-new state. The expected deterioration is assumed to degrade linearly in time from the initial condition of $r_0 = 100\%$ down to the failure level of $s = 0\%$. The deterioration is assumed to be distributed according to a stationary gamma process with shape function $v(t) = at$ and scale parameter u. It is convenient to rewrite the probability density function of $X(t)$ in Equations (24) and (27) using the parameterisation

$$a = \frac{\mu^2}{\sigma^2}, \quad b = 1, \quad u = \frac{\mu}{\sigma^2}$$

as follows:

$$f_{X(t)}(x) = \text{Ga}\left(x \,\middle|\, \frac{\mu^2}{\sigma^2}t, \frac{\mu}{\sigma^2}\right) \tag{28}$$

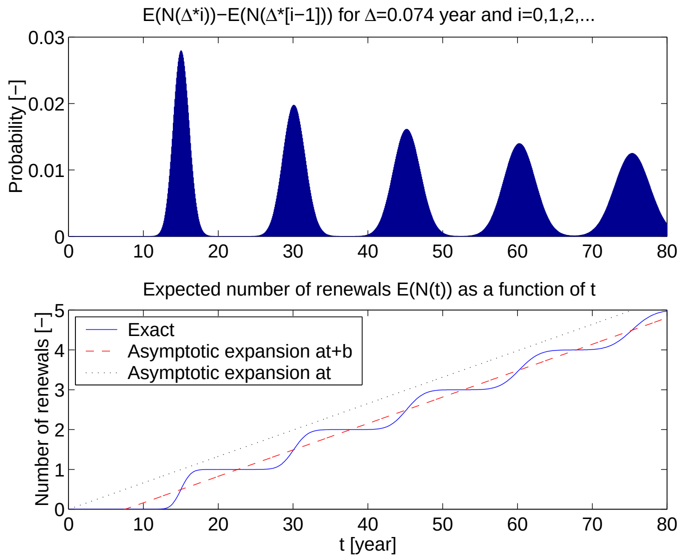

Figure 2: (Top) Probability of cylinder renewal per unit time of length $\Delta = 0.074$ year; (Bottom) Expected cumulative number of cylinder renewals.

for $\mu, \sigma > 0$ with

$$\mathbb{E}(X(t)) = \mu t, \quad \text{Var}(X(t)) = \sigma^2 t.$$

Due to the stationarity of the above deterioration process, both the mean value and the variance of the deterioration are linear in time. By using Equation (26), the cumulative lifetime distribution can then be rewritten as:

$$F(t) = \mathbb{P}(X(t) > y) = \frac{\Gamma([\mu^2 t]/\sigma^2, [y\mu]/\sigma^2)}{\Gamma([\mu^2 t]/\sigma^2)}, \tag{29}$$

where $y = r_0 - s$. For the hydraulic cylinder at hand, the time at which the expected condition equals the failure level is 15 years with parameters $\mu = 6.67$ and $\sigma = 1.81$.

A useful property of the gamma process with stationary increments is that the gamma density in Equation (28) transforms into an exponential density if

$$t = \left(\frac{\sigma}{\mu}\right)^2 = 0.074.$$

When the unit-time length is chosen to be $(\sigma/\mu)^2$, the increments of deterioration are exponentially distributed with mean σ^2/μ and the probability of failure in unit time i reduces to a shifted Poisson distribution (see, e.g., van Noortwijk et al., 1995):

$$p_i = \mathbb{P}(T = i) = \frac{1}{(i-1)!}\left[\frac{y\mu}{\sigma^2}\right]^{i-1} \exp\left\{-\frac{y\mu}{\sigma^2}\right\} \tag{30}$$

for $i = 1, 2, 3, \ldots$ Note that the shifted Poisson distribution is defined for $i = 1, 2, 3, \ldots$ rather than for $i = 0, 1, 2, \ldots$ The mean and variance of the shifted Poisson distribution are

$$\mathbb{E}(T) = 1 + \frac{y\mu}{\sigma^2}, \quad \text{Var}(T) = \frac{y\mu}{\sigma^2}.$$

The unit time for which the increments are exponentially distributed facilitates the algebraic manipulations considerably.

Using Equation (10), the probability of a cylinder renewal per unit time of length $\Delta = 0.074$ year is determined for every unit time over an eighty-year design life and displayed in Figure 2 (top). Again, because of the very small value of the first non-zero probability $p_1 = 4.2 \times 10^{-89}$, De Pril's recursion formula is unstable and we use the recursive convolution summation instead. For a bounded time horizon of eighty years, the expected cumulative number of cylinder replacements is determined using Equation (2) as well as approximated with the asymptotic expansions (14) and (16). In Figure 2 (bottom), we can clearly see that the asymptotic expansion including the constant term (16) is a much better approximation than the asymptotic expansion excluding the constant term (14).

4 CONCLUSIONS

This paper presents useful computational techniques to determine the probabilistic characteristics of a discrete renewal process. It includes methods to compute the probability distribution, expected value and variance of the number of renewals over a bounded time horizon, the asymptotic expansions for the expected value and variance of the number of renewals over an unbounded time horizon, and the extension of the discrete renewal model with the possibility of zero renewal times. The latter is necessary when approximating a continuous-time renewal processes with a discrete-time renewal process. For example, for discretising a continuous-time gamma-process stress-strength model, an upper-bound approximation of the cumulative distribution function is proposed. The advantage of applying the upper-bound (or lower-bound) approximation is that the only information needed is the cumulative distribution function of the lifetime. For the gamma process, the cumulative distribution function of the lifetime is available in explicit form (as opposed to the probability density function and the moments).

In order to compute the convolution of an arithmetic distribution, the traditional recursive convolution summation is preferred over De Pril's recursion formula known from the theory of insurance risk and ruin. The computational techniques presented in this paper can be used to model the initial transient phase of a renewal process (with, e.g., the recursive equation for the expected number of renewals) as well as the later equilibrium phase when the transients have passed (with, e.g., the asymptotic expansions for the expected number of renewals).

REFERENCES

Abdel-Hameed, M., 1975. A gamma wear process. *IEEE Transactions on Reliability*, 24(2):152–153.

De Pril, N., 1985. Recursions for convolutions of arithmetic distributions. *ASTIN Bulletin*, 15(2):135–139.

Dickson, D.C.M., 2005. *Insurance Risk and Ruin*. Cambridge: Cambridge University Press.

Feller, W., 1949. Fluctuation theory of recurrent events. *Transactions of the American Mathematical Society*, 67 (1):98–119.

Feller, W., 1950. *An Introduction to Probability Theory and its Applications; Volume 1*. New York: John Wiley & Sons.

Frangopol, D.M., Kallen, M.J., & van Noortwijk, J.M., 2004. Probabilistic models for life-cycle performance of deteriorating structures: Review and future directions. *Progress in Structural Engineering and Materials*, 6(4):197–212.

Heutink, A., van Beek, A., van Noortwijk, J.M., Klatter, H.E., & Barendregt, A., 2004. Environment-friendly maintenance of protective paint systems at lowest costs. In *XXVII FATIPEC Congress, 19-21 April 2004, Aix-en-Provence, France*, pages 351–364. Paris: AFTPVA.

Karlin, S. & Taylor, H.M., 1975. *A First Course in Stochastic Processes;* Second Edition. San Diego: Academic Press.

Smith, W.L., 1954. Asymptotic renewal theorems. *Proceedings of the Royal Society of Edinburgh, Section A (Mathematical and Physical Sciences)*, 64:9–48.

Smith, W.L., 1958. Renewal theory and its ramifications. *Journal of the Royal Statistical Society, Series B (Methodological)*, 20(2):243–302.

Sundt, B. & Dickson, D.C.M., 2000. Comparison of methods for evaluation of the n-fold convolution of an arithmetic distribution. *Bulletin of the Association of Swiss Actuaries*, pages 129–140.

van Noortwijk, J.M., 1998. Optimal replacement decisions for structures under stochastic deterioration. In Nowak, A., (ed.), *Proceedings of the Eighth IFIP WG 7.5 Working Conference on Reliability and Optimization of Structural Systems, Kraków, Poland, 11-13 May 1998*, pages 273–280. Ann Arbor: University of Michigan.

van Noortwijk, J.M., 2003. Explicit formulas for the variance of discounted life-cycle cost. *Reliability Engineering and System Safety*, 80(2):185–195.

van Noortwijk, J.M., Cooke, R.M., & Kok, M., 1995. A Bayesian failure model based on isotropic deterioration. *European Journal of Operational Research*, 82(2):270–282.