

# Gamma process model for time-dependent structural reliability analysis

M.D. Pandey

*Department of Civil Engineering, University of Waterloo, Waterloo, Ontario, Canada*

J.M. van Noortwijk

*HKV Consultants, Lelystad, Netherlands, and Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, Delft, Netherlands*

**ABSTRACT:** The paper presents a stochastic gamma process model to account for both sampling and temporal variability associated with a deterioration process that increases risk of failure of an aging structure. The proposed method is more versatile than the random-variable deterioration rate model commonly used in the structural reliability literature. The reason being that the random rate model cannot capture temporal variability associated with evolution of degradation. The paper presents an intuitive interpretation and a comparison of assumptions associated with the random variable and gamma process deterioration models. Numerical results presented in the paper are expected to improve the general understanding of time-dependent reliability analysis for maintenance planning and optimization.

## 1 INTRODUCTION

The time-dependent reliability analysis to model age-related deterioration is necessary to support strategic planning and optimization of civil engineering infrastructure systems. In the modelling of deterioration, two types of uncertainties are encountered, namely, sampling and temporal uncertainties. The sampling uncertainty means that parameters of the deterioration process vary from sample to sample, and uncertainty associated with the evolution or progression of deterioration over time is referred to as temporal uncertainty.

In engineering mechanics, the form of deterioration law, such as a linear or power law in time, is often known through experimental analysis and mathematical modelling. The sampling uncertainty can be captured by randomizing the parameters of the deterioration law, whereas its uncertain evolution over time should ideally be modelled as a stochastic process.

In structural reliability estimation, the deterioration of structures is usually modelled through random variables (such as the deterioration rate) and the computation is based on the First-Order Reliability Method (FORM). Such models, referred to as Random Variable (RV) deterioration model, do not account for temporal uncertainty in a formal mathematical sense. The paper presents a general model of deterioration based on the gamma process (GP), which is a stochastic process with independent non-negative increments having a gamma distribution (Figure 1). In contrast with the Brownian motion with drift, also called the Wiener process, the gamma process is monotonically increasing and positive, which makes it ideal for modelling deterioration processes.

Although the use of stochastic processes for modelling deterioration has been advocated, an intuitive explanation to draw a clear distinction between the features of the random variable versus the random process approach is not articulated in the engineering literature. The paper presents a comparative assessment and interpretation of the random variable and gamma process models for time-dependent reliability analysis in a simple setting to emphasize the additional value of information gained through the stochastic modelling.

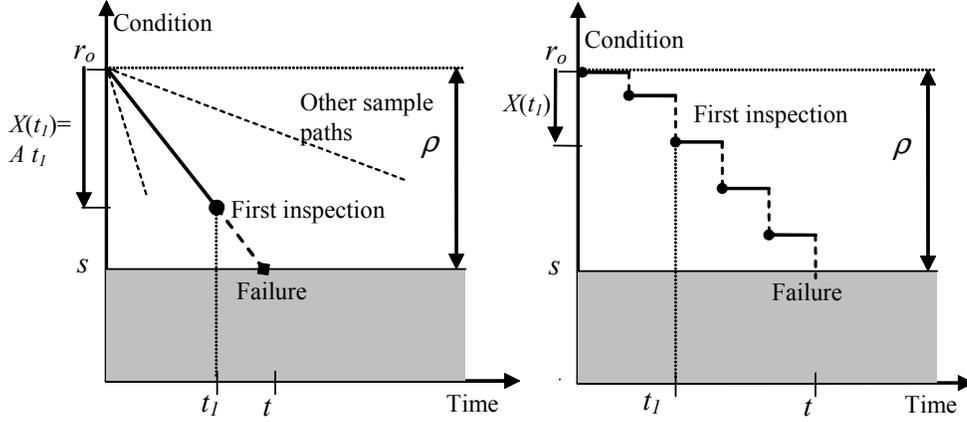


Figure 1: Models of deterioration: (a) RV model, and (b) GP model

## 2 RANDOM VARIABLE DEGRADATION MODEL

### 2.1 Model Definition

For the sake of clarity, consider that the deterioration of resistance follows a linear model given as

$$R(t) = r_0 - X(t) = r_0 - At \quad (1)$$

where  $r_0$  is the initial resistance,  $A$  is the random deterioration rate, and  $t$  is the time interval (or age). A common approach in time-dependent structural reliability analysis is to randomize the deterioration rate,  $A$ , and assign it a probability distribution based on some available data and expert judgment.

The randomization of the degradation rate reflects its variability in a large population of similar components (in a similar manner as the variability in lifetimes). The failure of a component is defined as the down-crossing of resistance below an applied stress  $s$ , and the probability of failure can be estimated from the limit state function:

$$R(t) - s \leq 0 = (r_0 - s) - X(t) \leq 0 = \rho - At \leq 0 \quad (2)$$

where  $\rho = (r_0 - s)$  denotes the design margin or a deterioration threshold. Failure is defined as the event at which the cumulative amount of deterioration  $X(t)$  exceeds the deterioration threshold  $\rho$ . The threshold  $\rho$  is taken as a constant for simplicity of discussion. We introduce the following notation to define the mean, variance and coefficient of variation (COV) of a random variable  $X$ :

$$E[X] = \mu_X, \quad \text{VAR}[X] = \sigma_X^2, \quad \text{and} \quad \text{COV}[X] = v_X = \frac{\sigma_X}{\mu_X} \quad (3)$$

The mean, variance and the COV of deterioration at time  $t$  are given as

$$\mu_{At} = \mu_A t, \quad \sigma_{At}^2 = \sigma_A^2 t^2, \quad \text{and} \quad v_{At} = \frac{\sigma_{At}}{\mu_{At}} = v_A \quad (4)$$

where  $\mu_A$ ,  $\sigma_A$  and  $v_A$  denote the mean, standard deviation and COV of the deterioration rate, respectively. From Equation (2), the cumulative probability distribution function of the lifetime,  $T$ , can be simply written as

$$F_T(t) = P[T \leq t] = P[(\rho / A) \leq t] = P[A \geq (\rho / t)] \quad (5)$$

Depending on the probability distribution of  $A$ , the life time distribution can be derived analytically or computed numerically using FORM or simulation.

## 2.2 Remarks

Although the degradation model in Equation (1) can be considered as a stochastic process in a technical sense, a sample path of deterioration of a specific component remains fixed over its entire lifetime. In other words, the temporal variability associated with the evolution of the deterioration process of a sample path over time is ignored. It is also reflected from Equation (4) that the COV of deterioration is constant over time. In fact, the future sample path in the linear deterioration model becomes completely deterministic after a single inspection at a time  $t_i$  quantifying the amount of the deterioration, as shown in Figure 1. It means that a single inspection can determine the remaining lifetime of the component without any uncertainty. In principle, if there are  $n$  random variables in the deterioration law,  $n$  number of inspections will determine the remaining lifetime of a component.

The RV deterioration model is implicit in several studies that apply FORM for time-dependent reliability analysis (Pandey 1998). In the condition assessment and rehabilitation planning of existing structures, the uncertainty associated with the evolution of degradation over time is an important consideration. Since the RV model is not adequate to model temporal variability associated with deterioration, we present a more formal stochastic gamma process model to overcome this limitation.

## 2.3 Illustration

For sake of a consistent comparison between the random variable and the gamma process model we assume that the degradation rate,  $A$ , is a gamma distributed random variable and the damage threshold  $\rho$  has a fixed deterministic value. The probability density function of the deterioration rate is given as

$$f_A(a) = \frac{\delta^\eta}{\Gamma(\eta)} a^{\eta-1} e^{-\delta a} = \text{Ga}(a | \eta, \delta), \quad a > 0, \quad (6)$$

where  $\eta$  and  $\delta$  are the shape and scale parameter of the gamma probability distribution, respectively. Note that we adopt  $\text{Ga}(x|\eta, \delta)$  as a compact notation for the density function of a gamma distributed random variable  $X$  with shape and scale parameters of  $\eta$  and  $\delta$ , respectively. The cumulative gamma distribution function is denoted as

$$\text{GA}(y|\eta, \delta) = \int_0^y \text{Ga}(x | \eta, \delta) dx \quad (7)$$

The mean and variance of the degradation rate are given as

$$\mu_A = \frac{\eta}{\delta} \quad \text{and} \quad \sigma_A^2 = \frac{\eta}{\delta^2} \quad (8)$$

Conversely, the shape and scale parameters can be related to the mean and COV,  $\nu_A = \sigma_A/\mu_A$ , as follows:

$$\eta = \frac{1}{(\nu_A)^2} \quad \text{and} \quad \delta = \frac{\eta}{\mu_A} = \frac{1}{\mu_A \nu_A^2} \quad (9)$$

The cumulative damage in time interval  $(0, t]$ ,  $X(t) = At$ , is also a gamma distributed random variable with the density function:

$$f_{X(t)}(x) = \text{Ga}\left(x \left| \eta, \frac{\delta}{t} \right.\right) \quad (10)$$

In this example, we assume  $\mu_A = 2$  units/year with a COV of  $\nu_A = 0.4$ , and the threshold  $\rho = 100$  units. The distribution parameters are  $\eta = 6.25$  and  $\delta = 3.125$ . The probability density functions

of deterioration in different time intervals ranging from 10 – 50 years are shown in Figure 2, which are calculated using Equation (10).

Since the deterioration rate is gamma distributed, the lifetime ( $T=\rho/A$ ) follows an inverted gamma distribution with the following density function:

$$f_T(t) = \frac{(\rho\delta)^\eta}{\Gamma(\eta)} \left(\frac{1}{t}\right)^{\eta+1} e^{-\rho\delta/t} \quad (11)$$

and the cumulative distribution function is written as

$$F_T(t) = 1 - \text{GA}\left(\frac{1}{t} \mid \eta, \rho\delta\right) \quad (12)$$

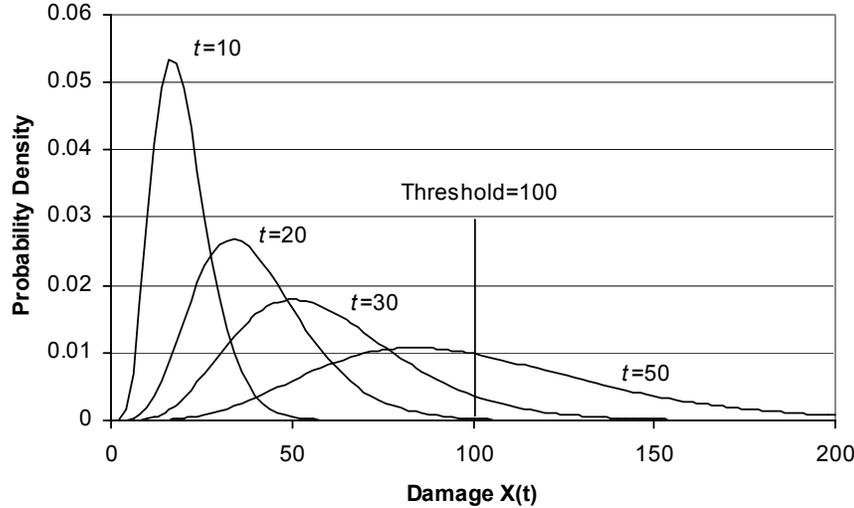


Figure 2: Probability density functions of deterioration  $X(t)$  in RV model

### 3 STOCHASTIC GAMMA PROCESS DETERIORATION MODEL

#### 3.1 Background

A key issue in time-dependent reliability analysis is the modeling of the deterioration process that typically increases in an uncertain manner over the life of a structure. In structural engineering, a distinction is made between a structure's resistance (e.g. the crest-level of a dike) and its applied stress (e.g. the water level to be withstood). A failure may then be defined as the event in which the deteriorating resistance drops below the stress. Maintenance management mainly deals with condition failure rather than structural failure (collapse). The uncertain deterioration can be regarded as a stochastic process, and the associated uncertainty can be represented by the normal distribution. This probability distribution has been used for modelling the exchange-value of shares and the movement of small particles in fluids and air. A characteristic feature of this model, also denoted by the Brownian motion with drift, is that a structure's resistance alternately increases and decreases, like the exchange-value of a share. For this reason, the Brownian motion is inadequate for modelling deterioration. For example, a dike of which the height is subject to a Brownian deterioration can, according to the model, spontaneously rise up, which of course cannot occur in practice.

The gamma process is ideally suited to model gradual damage that monotonically accumulates over time, such as wear, corrosion, erosion, and creep of materials, which are common causes of failure of engineering components (Abdel-Hameed, 1975).

### 3.2 Gamma Process Model for Deterioration

The gamma process belongs to a general class of stochastic processes, referred to as the Markov process. In the gamma process, increments are independent and non-negative random variables (e.g. metal loss due to corrosion) having a gamma distribution with an identical scale parameter and a time dependent shape function. The other examples of processes with independent increments are the Brownian motion with drift having Gaussian increments and the compound Poisson process.

In the gamma process model, the cumulative deterioration at time  $t$  follows a gamma distribution with the shape function,  $\lambda(t) > 0$ , and the scale parameter,  $\beta$  is a constant. To reflect the monotonic nature of deterioration over time, the shape function,  $\lambda(t)$ , must be an increasing function of time. The probability density function is given as

$$f_{X(t)}(x) = \frac{\beta^{\lambda(t)}}{\Gamma(\lambda(t))} x^{\lambda(t)-1} e^{-\beta x} = Ga(x | \lambda(t), \beta) \quad (13)$$

The damage increment from time  $t_1$  to  $t_2$ , given as  $[X(t_2) - X(t_1)]$ , is a non-negative quantity that is independent of the cumulative deterioration at time  $t_1$ ,  $X(t_1)$ . The increment is gamma distributed with shape parameter  $[\lambda(t_2) - \lambda(t_1)]$  and scale parameter  $\beta$ . The fact that the damage increment from any state  $X(t)$  is an independent random variable is a consequence of the properties of the gamma process. The gamma distribution is an infinitely divisible distribution, because of which the increments and their cumulative sum are gamma distributed. The sample paths of the gamma process are discontinuous and monotone, as shown in Figure 1.

The mean, variance and COV of the cumulative deterioration at time  $t$  are given as

$$\mu_{X(t)} = \frac{\lambda(t)}{\beta}, \quad \sigma_{X(t)}^2 = \frac{\lambda(t)}{\beta^2}, \quad \text{and} \quad \nu_{X(t)} = \frac{1}{\sqrt{\lambda(t)}} \quad (14)$$

A comparison of Equations (4) and (14) shows that COV of deterioration in the random variable model is time invariant, but it is a time dependent function in the gamma process model. The cumulative distribution function of the lifetime can be derived as

$$F_T(t) = P[X(t) \geq \rho] = 1 - P[X(t) < \rho] = 1 - GA[\rho | \lambda(t), \beta] \quad (15)$$

It should be noted that the probability density function,  $f(t) = dF(t)/dt$ , has no closed form expression, though it can be computed numerically.

### 3.3 Illustration

For the sake of a consistent comparison with the RV model, we consider deterioration as a stationary gamma process with  $\lambda(t) = \alpha t$ . It implies that the average deterioration increases linearly with time, though the gamma process is not restricted to a linear model. As a matter of fact, any shape function  $\lambda(t)$  suffices, as long as it is a non-decreasing, right continuous, and real-valued function.

The data given in the example of Section 2.3 are used to calibrate the parameters of the gamma process. The COV of deterioration is time dependent in GP model (Equation 14), and given as

$$\nu_{X(t)} = \frac{1}{\sqrt{\alpha t}} \quad (16)$$

To calibrate the gamma process, we take  $\nu_{X(t)} = \nu_A = 0.4$  at the time at which the expected deterioration exceeds the failure threshold, that is, at  $t_F = 50$  years. The shape parameter is calculated as  $\alpha = \eta/t_F = 6.25/50 = 0.125$ . The scale parameter is obtained by matching the mean damage rate in both RV and GP models, which leads to  $\beta = \delta/t_F = 3.125/50 = 0.0625$ . The probability density functions of cumulative damage at different time intervals are shown in Figure 3.

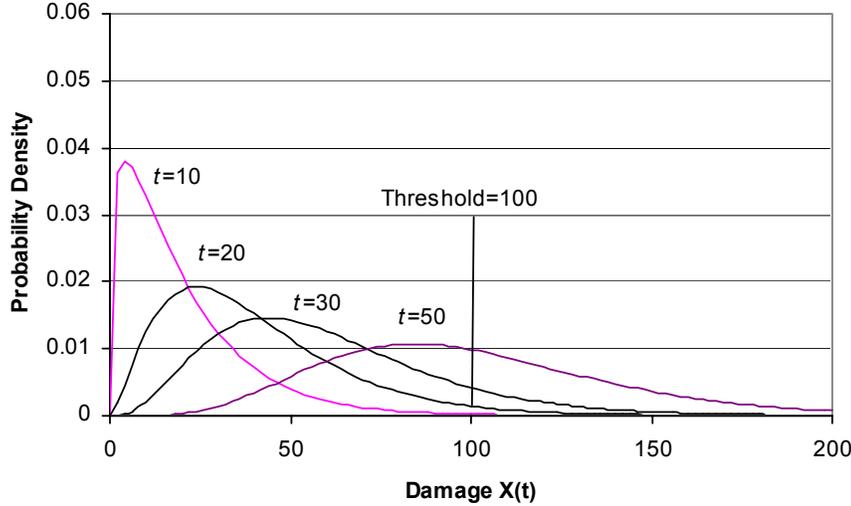


Figure 3: Probability density functions of deterioration  $X(t)$  in the gamma process model

### 3.4 Comparison of the GP and RV Models

As seen from Figure 1, the main difference between the GP model and the RV model is that the sample paths of the gamma process are discontinuous and monotone, whereas the sample paths of the latter approach are straight lines for a linear deterioration law. According to theory of the gamma process, one inspection reveals only one observed increment which can be used to predict the probability distribution of future deterioration. According to the random-variable degradation model, however, one inspection already fixes the future deterioration beforehand. Although the RV model can be a good approximation, one should be careful as soon as inspections are involved. Details of inspection models based on the gamma process are presented elsewhere by van Noortwijk et al. (1995, 1997).

In Figure 4, the lifetime density of the RV model (Equation 11) is compared with that computed for the GP model by discretizing the cumulative distribution given in Equation (15). The lifetime cumulative distribution functions are compared in Figure 5(a), which shows that the tails of the two distributions are different. This difference is seen more clearly in Figure 5(b), where the survival curves for both models are plotted on a log scale. It is interesting to note that survival probabilities in advanced ages ( $t > 50$ ) in the GP model are much less than those obtained from the RV model, in spite of the fact that mean deterioration rate is the same in both models. The pessimistic prediction of the GP model is attributed to uncertainty associated with evolution of deterioration in form of independent gamma increments. In contrast, the deterioration rate in the RV model is fully correlated over the lifetime, which possibly results in the overestimation of survival probabilities.

Another way to compare the GP and RV models is to study the probability of realizing different levels of condition (or remaining resistance margin) during the lifetime, such as a case shown in Figure 6. In case of the GP model ( $\nu_A = 0.4$ ), at the origin ( $\rho = 100$ ,  $t = 0$ ) the lifetime corresponding to 5th, 50th and 95th percentiles (5%, 50% and 95% quantiles) are estimated as 22, 50 and 90 years, respectively (Figure 6a), whereas the RV model leads to lifetime estimates of 30, 50 and 110 years for 5th, 50th and 95th percentiles, respectively. These bounds are computed from Equations (12) and (15) by decreasing  $\rho$  from 100 to 0. It can also be computed by using Equations (10) and (13) and determining percentiles for different values of  $t$ . The discrepancy between the GP and the RV models is amplified substantially, when the COV of deterioration rate is increased to  $\nu_A = 1$  (Figure 7). The upper bound curve (95% percentile) in the RV model is highly skewed and it results in unreasonably large values of the lifetime (Figure 7b), whereas the GP model provides reasonable bounds on the lifetime. An arbitrary increase in the uncertainty associated with the deterioration process is handled by the gamma process model in a logical manner, whereas the RV model appears to provide unreasonable estimates of the lifetime.

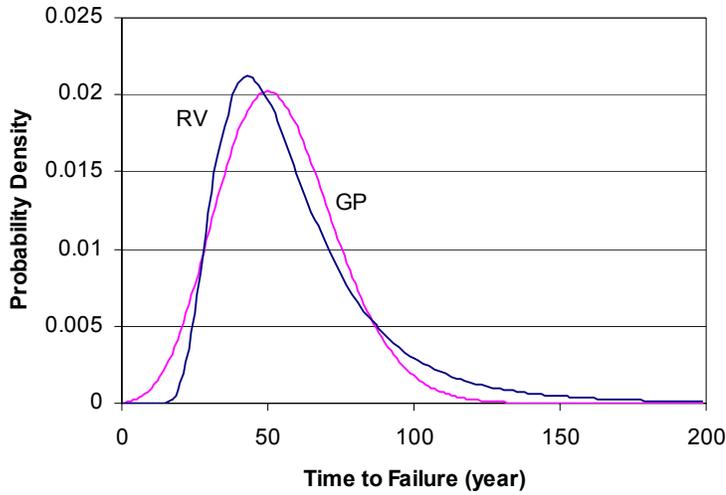


Figure 4: Comparison of lifetime probability density functions for RV and GP models

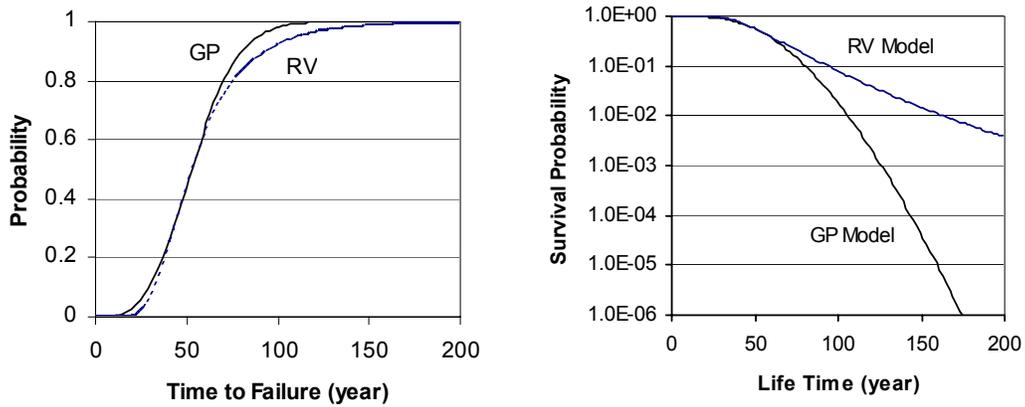


Figure 5: Comparison of lifetime distribution obtained from RV and GP models: (a) Cumulative distribution functions, and (b) Survival curves

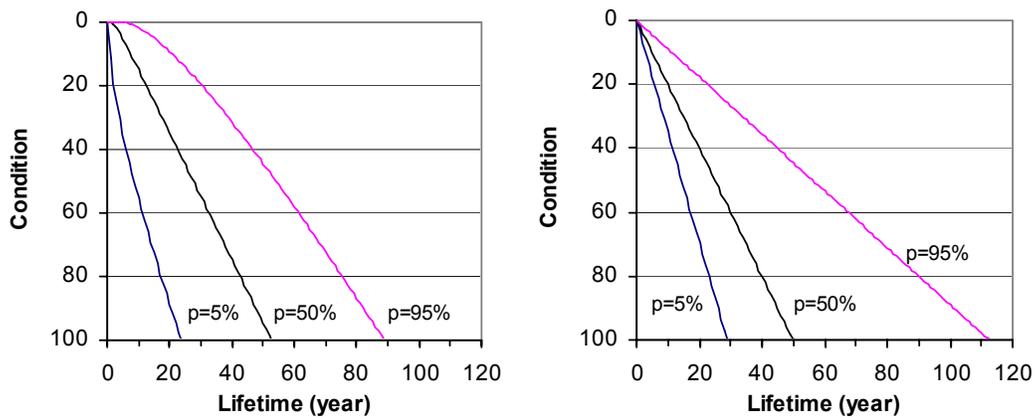


Figure 6: Probability of realizing different condition during the lifetime for  $\nu_A = 0.4$ : (a) GP model, and (b) RV model

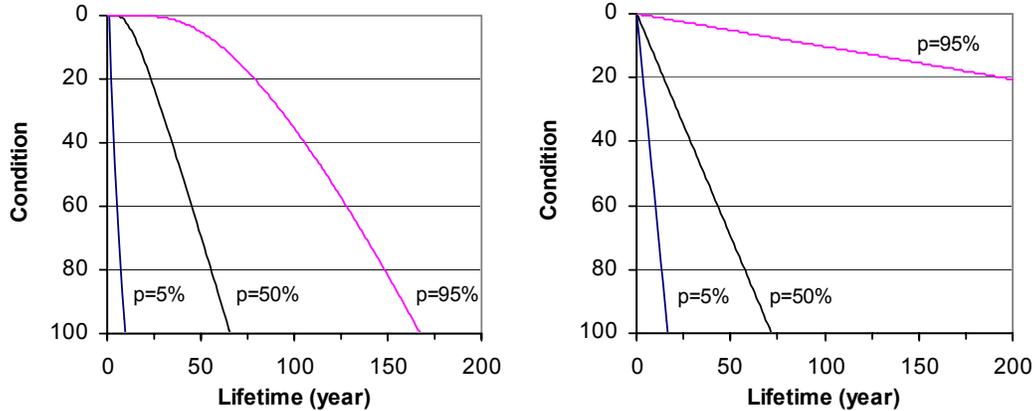


Figure 7: Probability of realizing different condition levels during the lifetime for  $\nu_A = 1$ : (a) GP model, and (b) RV model

#### 4 CONCLUSIONS

The paper presents a stochastic gamma process model to account for both sampling and temporal variability associated with a deterioration process that increases the probability of failure with aging of the structure. The proposed gamma process model is more versatile than the random-variable damage rate model commonly used in the structural reliability literature. The reason being that the random rate model cannot capture temporal variability associated with the evolution of degradation. For example, the coefficient of variation of cumulative deterioration in the random variable model is time independent, whereas the gamma process has an explicit dependence over time. A consequence is that the random variable model overestimates the probability of survival over a long term horizon. The temporal variability associated with the deterioration process is incorporated by the gamma process model in a more logical manner, and it provides more reasonable estimates of lifetime. A comparative evaluation of the gamma process and random variable models presented in the paper would contribute to increased use of stochastic processes in the modeling of structural deterioration.

#### 5 REFERENCES

- Abdel-Hameed, M. (1975). A gamma wear process. *IEEE Transactions on Reliability*, 24(2), 152-153.
- Barlow, R.E. and Proschan, F. 1965. *Mathematical Theory of Reliability*. John Wiley & Sons Inc., New York, NY.
- Pandey, M. (1998). Probabilistic models for condition assessment of oil and gas pipelines. *Int. J. Non-Destructive Testing and Evaluation*, 31(5), 349-358.
- van Noortwijk, J.M., Cooke, R.M. and Kok, M. (1995). A Bayesian failure model based on isotropic deterioration. *European Journal of Operational Research*, 82(2), 270-282.
- van Noortwijk, J.M., Kok, M., and Cooke, R.M. (1997). Optimal maintenance decisions for the sea-bed protection of the Eastern-Scheldt barrier. In R. Cooke, M. Mendel, and H. Vrijling, editors, *Engineering Probabilistic Design and Maintenance for Flood Protection*, pages 25-56. Dordrecht: Kluwer Academic Publishers.

#### 6 ACKNOWLEDGEMENTS

The first author (MDP) would like to acknowledge a partial financial support for this study provided by the Delft University of Technology, Delft, The Netherlands. The authors are also thankful to Mr. M.J. Kallen of Delft University of Technology for several helpful discussions.