A stochastic deterioration process for time-dependent reliability analysis*

J.M. van Noortwijk
HKV Consultants, Lelystad, Netherlands, and Delft University of Technology, Delft, Netherlands

M.D. Pandey
Department of Civil Engineering, University of Waterloo, Waterloo, Canada

Keywords: Deterioration, Reliability, Gamma process, Stochastic analysis, Probability

ABSTRACT: The paper presents a stochastic gamma process model to account for both population (i.e., sampling) and temporal variability associated with a degradation process that typically increases the probability of failure with the aging of a structure. The proposed method is more versatile than the random-variable degradation rate model commonly used in the structural reliability literature. The reason being that the random rate model cannot capture temporal variability associated with evolution of degradation. The paper also describes two methods for estimating parameters of the gamma process to facilitate its practical engineering applications.

1 INTRODUCTION

The condition assessment of a structure reveals its current state in relation to the structure's resistance to applied load effects. Due to aging-related deterioration, the current condition of a structure tends to be different from the original design and construction. A key issue, however, in planning future rehabilitation is to predict the trend of future degradation of resistance. The evolution of degradation tends to be uncertain in most engineering structures due to variability inherent in aging process, load effects and the operating environment. A rational approach to modelling temporal variability associated with degradation processes is the main objective of the paper.

Because deterioration is uncertain over time, it should ideally be represented as a stochastic process. At first glance, it seems possible to represent the temporal variability in a deterioration process by the Brownian motion with drift. A characteristic feature of this stochastic process, however, is that a structure's resistance alternately increases and decreases. For this reason, the Brownian motion is inadequate in modelling deterioration which is monotone.

In order to model monotonic progression of a deterioration process, the paper proposes a stochastic gamma process model for the resistance of a structural component. The gamma process is a stochastic process with independent, non-negative increments having a gamma distribution with identical scale parameter and a time-dependent shape parameter.

*Published in M.A. Maes and L. Huyse, editors, *Proceedings of the Eleventh IFIP WG 7.5 Working Conference on Reliability and Optimization of Structural Systems, 2-5 November 2003, Banff, Canada*, pages 259-265. London: Taylor & Francis Group, 2004.

The paper presents a time-dependent reliability analysis method based on gamma processes, describes statistical estimation methods and highlights the advantages of the proposed method over the conventional approach.

2 RANDOM-VARIABLE DEGRADATION MODEL

The time-dependent reliability analysis typically involves a degradation model for the resistance as

$$R(t) = R_0 - At^b, \tag{1}$$

where R_0 is the initial resistance, A is the random rate of degradation, t is the age of the component, and b reflects a nonlinear trend of the degradation law. The randomisation of the degradation rate reflects its variability in a large population of similar components (in a similar manner as the variability in lifetimes). If we define failure of a component as the down-crossing of resistance below an applied stress S, the failure-limit state can be written as $R(t) - S = 0$. The lifetime distribution can then be simply obtained from the relation

$$t = \left[\frac{R_0 - S}{A}\right]^{1/b}. \tag{2}$$

The lifetime distribution can be derived in an analytical or numerical sense depending on the probability distribution of R_0, S and A. As an illustration, consider the initial resistance and stress to be deterministic variables denoted by r_0 and s, and linear degradation law ($b = 1$) with rate A as a gamma distributed random variable. Under these assumptions the lifetime distribution is an inverted gamma distribution. Another example is a lognormal distribution for A as used by Mori & Ellingwood (1994).

Although the degradation model in Eq. (1) can be considered as a stochastic process in a technical sense, the sample paths of the degradation of a component remain fixed (e.g., linear in case of $b = 1$)

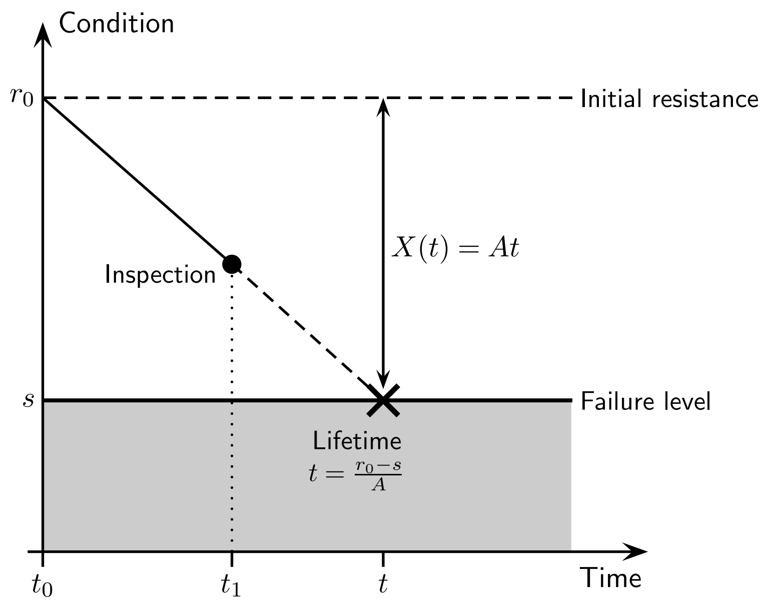

Figure 1: Random-Variable Degradation Model in case of a single inspection.

over its entire lifetime. In other words, the temporal variability associated with evolution of the degradation process of a sample path over time is not captured. In fact, the future sample path in the linear degradation model becomes completely deterministic after a single inspection that can quantify the extent of the degradation, as shown in Figure 1. Furthermore, the single inspection can also determine the remaining lifetime of the component without any uncertainty. In principle, if there are n unknowns in the degradation law, n number of inspections will determine the remaining lifetime of a component. The Random-Variable Degradation (RVD) model is implicit in several studies that apply the First-Order Reliability Method (FORM) for time-dependent reliability analysis. In the condition assessment and rehabilitation planning of existing structures, the uncertainty associated with the evolution of degradation over time is an important consideration for the optimisation of inspection and maintenance programs. Since the RVD model is not adequate to model temporal variability associated with degradation, we present a more formal Stochastic-Process Degradation (SPD) model to overcome this limitation. The proposed model is based on the theory of gamma processes as described in the next Section.

3 STOCHASTIC-PROCESS DEGRADATION MODEL

A difficulty in modelling time-dependent reliability is that the process of deterioration is uncertain over the life of a structure. In structural engineering, a distinction is made between a structure's resistance (e.g. the crest-level of a dike) and its applied stress (e.g. the water level to be withstood). A failure may then be defined as the event in which the deteriorating resistance drops below the stress. Maintenance management mainly deals with condition failure rather than structural failure (collapse).

The uncertain deterioration can be regarded as a stochastic process, and the associated uncertainty can be represented by the normal distribution. This probability distribution has been used for modelling the exchange-value of shares and the movement of small particles in fluids and air. A characteristic feature of this model, also denoted by the Brownian motion with drift (Karlin and Taylor 1975, Chapter 7), is that a structure's resistance alternately increases and decreases, like the exchange-value of a share. For this reason, the Brownian motion is inadequate in modelling deterioration which is monotone. For example, a dike of which the height is subject to a Brownian deterioration can, according to the model, spontaneously rise up, which of course cannot occur in practice.

In order to incorporate a monotonic increase in deterioration with age, we propose the gamma process as an ideal alternative (van Noortwijk et al. 1997). The gamma process is a stochastic process with independent, non-negative increments (e.g. the increments of crest-level decline of a dike) having a gamma distribution with an identical scale parameter. It implies that in case of a gamma deterioration, dikes can only decrease in height due to crest-level decline.

As far as the authors know, Abdel-Hameed (1975) was the first to propose the gamma process as a proper model for deterioration occurring random in time. In his two-page paper he called this stochastic process the "gamma wear process". An advantage of modelling deterioration processes through gamma processes is that the required mathematical calculations are relatively straightforward.

The gamma process is suitable to model gradual damage monotonically accumulating over time, such as wear, fatigue, corrosion, crack growth, erosion, consumption, creep, swell, degrading health index, et cetera. Other examples of the application of gamma processes are the theory of water storage by dams (Moran 1959, Chapter 4) and the theory of risk of ruin due to aggregate insurance claims (Dufresne et al. 1991). It should be noted, however, that dam storage and risk ruin models are inherently different from stress-strength models.

The mathematical definition of the gamma process is given as follows. Recall that a random quantity X has a gamma distribution with shape parameter $v > 0$ and scale parameter $u > 0$ if its probability density function is given by:

$$\text{Ga}(x|\,v,u) = \frac{u^v}{\Gamma(v)}\,x^{v-1}\exp\{-ux\}\,I_{(0,\infty)}(x),$$

where $I_A(x) = 1$ for $x \in A$ and $I_A(x) = 0$ for $x \notin A$, and $\Gamma(a) = \int_{t=0}^{\infty} t^{a-1}e^{-t}\,dt$ is the gamma function for $a > 0$. Furthermore, let $v(t)$ be a non-decreasing, right continuous, real-valued function for $t \geq 0$, with $v(0) \equiv 0$. The gamma process with shape function $v(t) > 0$ and scale parameter $u > 0$ is a continuous-time stochastic process $\{X(t),\ t \geq 0\}$ with the following properties:

1. $X(0) = 0$ with probability one;
2. $X(\tau) - X(t) \sim \text{Ga}(v(\tau) - v(t), u)$ for all $\tau > t \geq 0$;
3. $X(t)$ has independent increments.

Let $X(t)$ denote the deterioration at time t, $t \geq 0$, and let the probability density function of $X(t)$, in accordance with the definition of the gamma process, be given by

$$f_{X(t)}(x) = \text{Ga}\,(x\,|\,v(t), u\,) \tag{3}$$

with

$$E(X(t)) = \frac{v(t)}{u}, \quad \text{Var}(X(t)) = \frac{v(t)}{u^2}. \tag{4}$$

A component is said to fail when its deteriorating resistance, denoted by $R(t) = r_0 - X(t)$, drops below the stress s. We assume both the initial resistance r_0 and the stress s to be deterministic. Let the time at which failure occurs be denoted by the lifetime T. Due to the gamma distributed deterioration, Eq. (3), the lifetime distribution can then be written as:

$$\begin{aligned} F(t) &= \Pr\{T \leq t\} = \Pr\{X(t) \geq r_0 - s\} = \\ &= \int_{x=r_0-s}^{\infty} f_{X(t)}(x)\,dx = \frac{\Gamma(v(t), [r_0 - s]u)}{\Gamma(v(t))}, \end{aligned} \tag{5}$$

where $\Gamma(a,x) = \int_{t=x}^{\infty} t^{a-1}e^{-t}\,dt$ is the incomplete gamma function for $x \geq 0$ and $a > 0$. Using the chain rule for differentiation, the probability density function of the lifetime is

$$f(t) = \frac{\partial}{\partial t}\left[\frac{\Gamma(v(t), [r_0 - s]u)}{\Gamma(v(t))}\right] = \frac{\partial}{\partial \tilde{v}}\left[\frac{\Gamma(\tilde{v}, [r_0 - s]u)}{\Gamma(\tilde{v})}\right]\Bigg|_{\tilde{v}=v(t)} v'(t) \tag{6}$$

under the assumption that the shape function $v(t)$ is differentiable. The partial derivative in Eq. (6) can be calculated by the algorithm of Moore (1982). Using a series expansion and a continued fraction expansion, this algorithm computes the first and second partial derivatives with respect to x and a of the incomplete gamma integral

$$P(a,x) = \frac{1}{\Gamma(a)}\int_{t=0}^{x} t^{a-1}e^{-t}\,dt = \frac{\Gamma(a) - \Gamma(a,x)}{\Gamma(a)}.$$

Under the assumption of modelling the temporal variability in the deterioration in terms of a gamma process, the question which remains to be answered is how its expected deterioration increases over time. Empirical studies show that the expected deterioration at time t is often proportional to a power law:

$$v(t) = ct^b \tag{7}$$

for some physical constants $c > 0$ and $b > 0$. Some examples of expected deterioration according to a power law are the expected degradation of concrete due to corrosion of reinforcement (linear: $b = 1$; Ellingwood & Mori (1993)), sulfate attack (parabolic: $b = 2$; Ellingwood & Mori (1993)), diffusion-controlled aging (square root: $b = 0.5$; Ellingwood & Mori (1993)), and creep ($b = 1/8$; Çinlar et al. (1977)), and the expected scour-hole depth ($b = 0.4$; Hoffmans & Pilarczyk (1995) and van Noortwijk & Klatter (1999)). Because there is often engineering knowledge available about the shape of the expected deterioration in terms of the parameter b in Eq. (7), this parameter may be assumed constant. In the event of expected deterioration in terms of a power law, the parameters c and u yet must be assessed by using expert judgment and/or statistics. It should be noted that the gamma process is not restricted to using a power law for modelling the expected deterioration over time. As a matter of fact, any shape function $v(t)$ suffices, as long as it is a non-decreasing, right continuous, and real-valued function.

The main difference between the SPD model and the RVD model is that the sample paths of the former approach are discontinuous and monotone, whereas the sample paths of the latter approach are straight lines (for a linear degradation law). According to the gamma process, one inspection thus reveals only one observed increment which can be used to predict future deterioration. According to the random-variable degradation model, however, one inspection already fixes the future deterioration beforehand (see Figure 1). Although the RVD model can be very well used as an approximation, one should be careful as soon as inspections are involved. For inspection models based on the gamma process, see e.g. van Noortwijk et al. (1995, 1997).

4 STATISTICAL ESTIMATION

In order to apply the gamma process model to practical examples, statistical methods for the parameter estimation of gamma processes are required. A typical data set consists of inspection times t_i, $i = 1, \ldots, n$, where $0 = t_0 < t_1 < t_2 < \cdots < t_n$, and corresponding observations of the cumulative amounts of deterioration x_i, $i = 1, \ldots, n$, where $0 = x_0 \le x_1 \le x_2 \le \cdots \le x_n$. Consider a gamma process with shape function $v(t) = ct^b$ and scale parameter u. We assume that the value of the power b is known, but c and u are unknown. The two most common methods of parameter estimation, namely, Maximum Likelihood and Method of Moments, are discussed in this Section. Both methods for deriving the estimators of c and u were initially presented by Çinlar et al. (1977).

4.1 *Method of Maximum Likelihood*

The maximum-likelihood estimators of c and u can be obtained by maximising the logarithm of the likelihood function of the increments. The likelihood function of the observed deterioration increments $\delta_i = x_i - x_{i-1}$, $i = 1, \ldots, n$, is a product of independent gamma densities

$$\ell(\delta_1, \ldots, \delta_n | c, u) = \prod_{i=1}^{n} f_{X(t_i)-X(t_{i-1})}(\delta_i) = \prod_{i=1}^{n} \frac{u^{c[t_i^b - t_{i-1}^b]}}{\Gamma(c[t_i^b - t_{i-1}^b])} \delta_i^{c[t_i^b - t_{i-1}^b]-1} \exp\{-u\delta_i\}. \tag{8}$$

It follows that the maximum-likelihood estimator of u is $\hat{u} = \hat{c}t_n^b / x_n$, where $\hat{c}$ must be computed iteratively. Given the maximum-likelihood estimator of u, the expected deterioration at time t can then be written is

$$E(X(t)) = x_n \left[\frac{t}{t_n}\right]^b.$$

Because cumulative amounts of deterioration are measured, the last inspection contains the most information. This is confirmed by the fact that the expected deterioration at the last inspection (at time t_n) equals x_n; that is, $E(X(t_n)) = x_n$. By taking $\hat{u} = \hat{c}t_n^b/x_n$, the maximum-likelihood estimate of c can be obtained by solving the following equation for c:

$$
\begin{aligned}
\frac{\partial}{\partial c}\log\ell(\delta_1,\ldots,\delta_n|c) \;&=\\
&= \sum_{i=1}^{n}\frac{\partial}{\partial c}\left\{c[t_i^b - t_{i-1}^b]\log\left(\frac{ct_n^b}{x_n}\right) - \log\Gamma(c[t_i^b - t_{i-1}^b]) + \left(c[t_i^b - t_{i-1}^b] - 1\right)\log\delta_i - \frac{ct_n^b}{x_n}\delta_i\right\}\\
&= \sum_{i=1}^{n}\left\{[t_i^b - t_{i-1}^b]\left[\log\left(\frac{ct_n^b}{x_n}\right) + 1 - \psi(c[t_i^b - t_{i-1}^b]) + \log\delta_i\right] - \frac{t_n^b}{x_n}\delta_i\right\}\\
&= t_n^b\log\left(\frac{ct_n^b}{x_n}\right) + \sum_{i=1}^{n}[t_i^b - t_{i-1}^b]\left\{\log\delta_i - \psi(c[t_i^b - t_{i-1}^b])\right\} = 0, \qquad (9)
\end{aligned}
$$

where the function $\psi(a)$ is the derivative of the logarithm of the gamma function:

$$
\psi(a) = \frac{\Gamma'(a)}{\Gamma(a)} = \frac{\partial\log\Gamma(a)}{\partial a}
$$

for $a > 0$. It is called the digamma function and can be accurately computed using the algorithm developed by Bernardo (1976). Summarising, the maximum-likelihood estimates $\hat{c}$ and $\hat{u}$ can be solved from

$$
\hat{u} = \frac{\hat{c}t_n^b}{x_n}, \quad \sum_{i=1}^{n}[t_i^b - t_{i-1}^b]\left\{\psi(\hat{c}[t_i^b - t_{i-1}^b]) - \log\delta_i\right\} = t_n^b\log\left(\frac{\hat{c}t_n^b}{x_n}\right). \qquad (10)
$$

4.2 *Method of Moments*

Recall that the expected value and variance of the accumulated deterioration at calendar time t are given by

$$
E(X(t)) = \frac{ct^b}{u}, \quad \mathrm{Var}(X(t)) = \frac{ct^b}{u^2}. \qquad (11)
$$

When the power b is known, the non-stationary gamma process can be easily transformed to a stationary gamma process by performing a monotonic transformation from the clock or calender time t to the transformed or operational time $z(t) = t^b$. A stochastic process has stationary increments if the probability distribution of the increments $X(t+h) - X(t)$ depends only on h for all $t, h \geq 0$. Substituting the inverse time transformation $t(z) = z^{1/b}$ in Eq. (11) yields

$$
E(X(t(z))) = \frac{cz}{u}, \quad \mathrm{Var}(X(t(z))) = \frac{cz}{u^2}. \qquad (12)
$$

This results in a stationary gamma process with respect to the transformed time z. Similarly, the transformed inspection times are $z_i = t_i^b$, $i = 1,\ldots,n$. Let us further define the transformed times between inspections as $w_i = t_i^b - t_{i-1}^b$, $i = 1,\ldots,n$, and, for mathematical convenience,

$$
D_i = X_i - X_{i-1}, \quad Y_i = D_i - \frac{cw_i}{u}, \quad i = 1,\ldots,n. \qquad (13)
$$

The deterioration increment D_i has a gamma distribution with shape parameter cw_i and scale parameter u for all $i = 1, \ldots, n$, and the increments $D_1, \ldots, D_n$ are independent. Note that X_i, D_i and Y_i denote random quantities and x_i, δ_i and y_i the corresponding observations. For each i, the first and second moment of Y_i are

$$E(Y_i) = 0, \quad E(Y_i^2) = \frac{cw_i}{u^2}, \quad i = 1, \ldots, n. \tag{14}$$

For notational convenience, we introduce the following average rates of deterioration per unit of transformed time:

$$\bar{D} = \frac{\sum_{i=1}^n D_i}{\sum_{i=1}^n w_i}, \quad \bar{Y} = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n w_i}, \quad \bar{Y} = \bar{D} - \frac{c}{u}. \tag{15}$$

From Eqs. (14) and (15), it follows that

$$E(\bar{Y}) = 0, \quad E(\bar{Y}^2) = \frac{\sum_{i=1}^n E(Y_i^2)}{(\sum_{i=1}^n w_i)^2} = \frac{c}{u^2}\frac{\sum_{i=1}^n w_i}{(\sum_{i=1}^n w_i)^2} = \frac{c}{u^2}\frac{1}{\sum_{i=1}^n w_i}. \tag{16}$$

Now, one may calculate $E(\bar{D}) = c/u$ and Eq. (13) yields

$$\begin{aligned}
\sum_{i=1}^n (D_i - \bar{D}w_i)^2 &= \sum_{i=1}^n \left[D_i - \frac{cw_i}{u} - \left(\bar{D} - \frac{c}{u}\right) w_i\right]^2 = \\
&= \sum_{i=1}^n \left[Y_i - \bar{Y}w_i\right]^2 = \sum_{i=1}^n \left(Y_i^2 - 2Y_i\bar{Y}w_i + w_i^2\bar{Y}^2\right).
\end{aligned} \tag{17}$$

Because $E(Y_i) = 0$, the second term in the last sum can be rewritten as

$$E(Y_i\bar{Y}) = \frac{E(Y_i \sum_{j=1}^n Y_j)}{\sum_{i=1}^n w_i} = \frac{E(Y_i^2 + Y_i \sum_{i \neq j} Y_i)}{\sum_{i=1}^n w_i} = \frac{E(Y_i^2)}{\sum_{i=1}^n w_i} = \frac{c}{u^2}\frac{w_i}{\sum_{i=1}^n w_i}. \tag{18}$$

By taking expectations of both sides of Eq. (17) and by applying Eqs. (16-18), it follows that

$$E\left(\sum_{i=1}^n (D_i - \bar{D}w_i)^2\right) = \frac{c}{u^2}\left(\sum_{i=1}^n w_i - \frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n w_i}\right). \tag{19}$$

Finally, the method-of-moments estimates $\hat{c}$ and $\hat{u}$ can be solved from

$$\frac{\hat{c}}{\hat{u}} = \frac{\sum_{i=1}^n \delta_i}{\sum_{i=1}^n w_i} = \frac{x_n}{t_n^b} = \bar{\delta}, \quad \frac{\hat{c}}{\hat{u}^2}\left(\sum_{i=1}^n w_i - \frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n w_i}\right) = \sum_{i=1}^n (\delta_i - \bar{\delta}w_i)^2 \tag{20}$$

or, equivalently,

$$\frac{\hat{c}}{\hat{u}} = \frac{\sum_{i=1}^n \delta_i}{\sum_{i=1}^n w_i} = \frac{x_n}{t_n^b} = \bar{\delta}, \quad \frac{x_n}{\hat{u}}\left(1 - \frac{\sum_{i=1}^n w_i^2}{[\sum_{i=1}^n w_i]^2}\right) = \sum_{i=1}^n (\delta_i - \bar{\delta}w_i)^2. \tag{21}$$

Clearly, the method of moments leads to simple formulae for parameter estimation which can be easily computed. Note that the first equation in the maximum-likelihood estimation (10) is the same as the first equation in the method-of-moments estimation (21).

ACKNOWLEDGEMENT

The work reported in this paper was partly supported by the Civil Engineering Division of the Netherlands Ministry of Transport, Public Works, and Water Management.

5 CONCLUSIONS

The paper presents a stochastic gamma process model to account for both population (i.e., sampling) and temporal variability associated with a degradation process that increases the probability of failure with aging of the structure. The proposed method is more versatile than the random-variable damage rate model commonly used in the structural reliability literature. The reason being that the random rate model cannot capture temporal variability associated with the evolution of degradation. The paper also describes two methods for estimating parameters of the gamma process to facilitate its practical application. In particular, the method of moments is quite simple to use. The exposition of gamma processes presented in the paper would contribute to increased use of stochastic processes in the modelling of structural degradation.

REFERENCES

Abdel-Hameed, M. (1975). A gamma wear process. *IEEE Transactions on Reliability 24*(2), 152–153.

Bernardo, J.M. (1976). Algorithm AS 103: Psi (digamma) function. *Applied Statistics 25,* 315–317.

Çinlar, E., Z.P. Bažant, and E. Osman (1977). Stochastic process for extrapolating concrete creep. *Journal of the Engineering Mechanics Division 103*(EM6), 1069–1088.

Dufresne, F., H.U. Gerber, and E.S.W. Shiu (1991). Risk theory with the gamma process. *ASTIN Bulletin 21*(2), 177–192.

Ellingwood, B.R. and Y. Mori (1993). Probabilistic methods for condition assessment and life prediction of concrete structures in nuclear power plants. *Nuclear Engineering and Design 142*, 155–166.

Hoffmans, G.J.C.M. and K.W. Pilarczyk (1995). Local scour downstream of hydraulic structures. *Journal of Hydraulic Engineering 121*(4), 326–340.

Karlin, S. and H.M. Taylor (1975). *A First Course in Stochastic Processes;* Second Edition. San Diego: Academic Press.

Moore, R.J. (1982). Algorithm AS 187: Derivatives of the incomplete gamma integral. *Applied Statistics 31*, 330–335.

Moran, P.A.P. (1959). *The Theory of Storage*. London: Methuen & Co.

Mori, Y. and B.R. Ellingwood (1994). Maintaining: Reliability of concrete structures. I: Role of inspection/repair. *Journal of Structural Engineering 120*(3), 824–845.

van Noortwijk, J.M., R.M. Cooke, and M. Kok (1995). A Bayesian failure model based on isotropic deterioration. *European Journal of Operational Research 82*(2), 270–282.

van Noortwijk, J.M. and H.E. Klatter (1999). Optimal inspection decisions for the block mats of the Eastern-Scheldt barrier. *Reliability Engineering and System Safety 65*(3), 203–211.

van Noortwijk, J.M., M. Kok, and R.M. Cooke (1997). Optimal maintenance decisions for the seabed protection of the Eastern-Scheldt barrier. In R. Cooke, M. Mendel, and H. Vrijling (Eds.), *Engineering Probabilistic Design and Maintenance for Flood Protection*, pp. 25–56. Dordrecht: Kluwer Academic Publishers.