Cost-based criteria for obtaining optimal design decisions

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ABSTRACT: In life-cycle costing analyses, optimal design is usually achieved by minimising the expected value of the discounted costs. Apart from the expected value, the corresponding variance may be useful as well for estimating, for example, the uncertainty bounds of the calculated discounted costs. However, general explicit formulas for calculating the variance of the discounted costs over an unbounded time horizon are not available yet. In this paper, explicit formulas for this variance are presented. They can be easily implemented in software to optimise structural design and management. The use of the mathematical results is illustrated with some examples.

1 INTRODUCTION

In life-cycle costing analyses, optimal design is usually achieved by minimising the expected value of the discounted costs. There are three main cost-based criteria for comparing design and maintenance decisions over an unbounded time horizon: (i) the expected average costs per unit time (which are determined by averaging the costs over an unbounded horizon), (ii) the expected discounted costs over an unbounded horizon (which are determined by summing the present discounted values of the costs over an unbounded horizon), and (iii) the expected equivalent average costs per unit time (which are determined by averaging the discounted costs). From these three criteria, only the last two are appropriate for optimal design decisions, because the contribution of the initial investment cost is ignored for the first one. The notion of equivalent average costs relates to the notions of average costs and discounted costs in the sense that the equivalent average costs per unit time approach the average costs per unit time, as the discount rate tends to zero from above. All three cost-based criteria can be obtained in explicit form using the renewal theorem.

Apart from the expected value of the costs, the variance of these costs is also important to know. In this paper, new explicit formulas will be presented for both the variance of the discounted costs over an unbounded horizon and the equivalent long-term average variance of the costs per unit time. As with the expected costs, the formulas have been derived using the renewal theorem. It can be shown that the equivalent average variance of the costs per unit time approaches the average variance of the costs per unit time, as the discount rate tends to zero from above. In doing so, the equivalent average variance of the costs per unit time approaches a known result from renewal reward theory.

Use of the new formulas is illustrated by deriving an optimal age replacement strategy for a cylinder. For several age replacement intervals, both the expected value and the variance of the discounted costs are determined. Because the new formulas lead to relatively simple expressions, they can be easily implemented for the purpose of life-cycle costing. Examples of possible applications are the models of Frangopol (1999), Frangopol et al. (1999, 2000, 2001), Kuschel & Rackwitz (2000), Rackwitz (2000), Speijker et al. (2000), and Van Noortwijk et al. (1996, 1997, 1998, 2000). With these mathematical models, optimal design and maintenance decisions can be made under uncertain deterioration and reliability.

The outline of this paper is as follows. Three cost-based criteria for comparing design and maintenance decisions are listed in Section 2: the average costs per unit time, the discounted costs over an unbounded horizon and the equivalent average costs per unit time. Mathematical formulas for the expected value and the variance of the discounted costs over an unbounded horizon are derived in Section 3 and 4 respectively. These formulas are based on results of discrete-time renewal theory. In Section 5, similar results are obtained for continuous-time renewal processes. Section 6 contains three illustrations of the theory, and concluding remarks are made in Section 7.

2 COST-BASED CRITERIA

The purpose of this paper is to derive both the expected value and the variance of the discounted costs of maintenance for uncertain deterioration. These probabilistic characteristics are useful for the optimisation of design and maintenance. In the design phase, the initial cost of investment has to be balanced against the future cost of maintenance. In the application phase, the cost of inspection and preventive replacement have to be balanced against the cost of corrective replacement and failure.

Usually, maintenance is defined as a combination of actions carried out to restore a component or structure, or to "renew" it, to the initial condition. Inspections, repairs, replacements, and lifetime-extending measures are possible maintenance actions. Through lifetime-extending measures, the deterioration can be delayed so that failure is postponed and the component's lifetime is extended. Roughly, there are two types of maintenance: corrective maintenance (mainly after failure) and preventive maintenance (mainly before failure). Corrective maintenance can best be chosen if the cost arising from failure is low (like for instance replacing a burnt-out light bulb); preventive maintenance if this cost is high (like for instance heightening a dyke).

Since the planned lifetime of most structures is very long, maintenance decisions may be compared over an unbounded time horizon. According to Wagner (1975, Chapter 11), there are basically three cost-based criteria that can be used to compare maintenance decisions:

1. the *expected average costs per unit time*, which are determined by averaging the costs over an unbounded time horizon;
2. the *expected discounted costs over an unbounded time horizon*, which are determined by summing the (present) discounted values of the costs over an unbounded time horizon, under the assumption that the value of money decreases in time;
3. the *expected equivalent average costs per unit time*, which are determined by calculating the discounted costs per unit time.

The notion of equivalent average costs relates to the notions of average costs and discounted costs in the sense that the equivalent average costs per unit time approach the average costs per unit time, as the discount rate tends to zero from above. The cost-based criteria of discounted costs and equivalent average costs are most suitable for balancing the initial building cost optimally against the future maintenance cost. The criterion of average costs can be used in situations in which no large investments are made (like inspections) and in which the time value of money is of no consequence. Often, it is preferable to spread the costs of maintenance over time and to use discounting.

3 EXPECTED VALUE OF COST

Maintenance can often be modelled as a discrete renewal process, whereby the renewals are the maintenance actions that bring a component back into its original condition or "as good as new state". After each renewal we start (in a statistical sense) all over again. A discrete renewal process $\{N(n), n = 1,2,3,\ldots\}$ is a non-negative integer-valued stochastic process that registers the successive renewals in the time-interval $(0,n]$. Let the renewal times T_1, T_2, T_3, …, be non-negative, independent, identically distributed, random quantities having the discrete probability function

$$\Pr\{T_k = i\} = p_i, \quad i = 1,2,3,\ldots, \quad \textstyle\sum_{i=1}^{\infty} p_i = 1,$$

where p_i represents the probability of a renewal in unit i. We denote the cost associated with a renewal in unit time i by c_i, $i = 1,2,3,\ldots$ The above-mentioned three cost-based criteria will be discussed in more detail in the following subsections.

3.1 *Expected average costs per unit time*

The expected average costs per unit time are determined by simply averaging the costs over an unbounded horizon. They follow from the expected costs over the bounded horizon $(0,n]$, denoted by $E(K(n))$, which is a solution of the recursive equation

$$E(K(n)) = \sum_{i=1}^{n} p_i [c_i + E(K(n-i))] \tag{1}$$

for $n = 1,2,3,\ldots$ and $K(0) \equiv 0$. To obtain this equation, we condition on the values of the first renewal time T_1 and apply the law of total probability. The costs associated with occurrence of the event $T_1 = i$ are c_i plus the additional expected costs during the interval $(i,n]$, $i = 1,\ldots,n$. Using the discrete renewal theorem [see Feller (1950, Chapter 12 & 13) and Karlin & Taylor (1975, Chapter 3)], the expected *average costs per unit time* are

$$\lim_{n\to\infty} \frac{E(K(n))}{n} = \frac{\sum_{i=1}^{\infty} c_i p_i}{\sum_{i=1}^{\infty} i p_i} = \frac{E(c_I)}{E(I)}. \tag{2}$$

Let a renewal cycle be the time-period between two renewals, and recognise the numerator as the expected cycle costs and the denominator as the expected cycle length (mean lifetime). Eq. (2) is a well-known result from renewal reward theory [see e.g. Ross (1970, Chapter 3)]. The limit in Eq. (2) exists provided that the greatest common divisor of the integers $i = 1,2,3,\ldots$ for which $p_i > 0$ is equal to one. The simplest assumption assuring this is $p_1 > 0$. If $c_i \equiv 1$ for all $i = 1,2,3,\ldots$ in Eq. (2), then the expected *average number of renewals per unit time* is:

$$\lim_{n\to\infty} \frac{E(N(n))}{n} = \frac{1}{\sum_{i=1}^{\infty} i p_i} = \frac{1}{E(I)}$$

being the reciprocal of the mean lifetime.

3.2 *Expected discounted costs over an unbounded horizon*

Discounting expected costs over an unbounded horizon is based on the assumption that the value of money decreases with time. Since the future cost can be discounted to its present value on the basis of a discount rate, we can compare the value of money at different dates. In mathematical terms, the *(present) discounted value* of the cost c_n in unit time n is defined as $\alpha^n c_n$ with $\alpha = [1+(r/100)]^{-1}$ the *discount factor* per unit time and r% the *discount rate* per unit time, where $r > 0$. The decision-maker is indifferent to the cost c_n at time n and the costs $\alpha^n c_n$ at time 0. Therefore, the higher the discount rate, the more beneficial it is to postpone expensive maintenance actions.

The expected discounted costs over a bounded time horizon can be obtained with a recursive formula similar to that of the expected costs in Eq. (1). Again, we condition on the values of the first renewal time T_1 and apply the law of total probability. In this case, however, it is desirable to account for the discounted value of the renewal costs c_i plus the additional expected discounted costs in time-interval $(i,n]$, $i = 1,\ldots,n$. Hence, the expected discounted costs over the bounded horizon $(0,n]$, denoted by $E(K_\alpha(n))$, can be written as

$$E(K_\alpha(n)) = \sum_{i=1}^{n} \alpha^i p_i [c_i + E(K_\alpha(n-i))] \qquad (3)$$

for $n = 1,2,3,\ldots$ and $K_\alpha(0) \equiv 0$. By using Feller (1950, Chapter 13), the expected *discounted costs over an unbounded horizon* can be written as

$$\lim_{n\to\infty} E(K_\alpha(n)) = \frac{\sum_{i=1}^{\infty} \alpha^i c_i p_i}{1-\sum_{i=1}^{\infty} \alpha^i p_i} = \frac{E(\alpha^I c_I)}{1-E(\alpha^I)} = k(\alpha). \quad (4)$$

We recognise the numerator of Eq. (4) as the expected discounted cycle costs, while the denominator can be interpreted as the probability that the renewal process "terminates due to discounting". Such a renewal process is called a *terminating renewal process* since infinite inter-occurrence times can cause the renewals to cease. The inter-occurrence times Z_1, Z_2, …, of the imaginary terminating renewal process have the distribution

$$\Pr\{Z_k = i\} = \alpha^i p_i, \quad i = 1,2,3,\ldots,$$

and

$$\Pr\{Z_k = \infty\} = 1 - \sum_{i=1}^{\infty} \alpha^i p_i.$$

The expected number of imaginary "discounted renewals" over an unbounded horizon is

$$\lim_{n\to\infty} E(N_\alpha(n)) = \frac{\sum_{i=1}^{\infty} \alpha^i p_i}{1-\sum_{i=1}^{\infty} \alpha^i p_i} = \frac{\Pr\{Z_k < \infty\}}{\Pr\{Z_k = \infty\}}.$$

3.3 *Expected equivalent average costs per unit time*

The expected equivalent average costs per unit time relate to the notions of average costs and discounted costs. To determine this relation, we construct a new infinite stream of identical costs with the same present discounted value as the expected discounted costs over an unbounded time horizon $k(\alpha)$. This can be achieved easily by defining an infinite stream of costs at times $i = 0,1,2,\ldots$, which are all equal to $(1-\alpha)k(\alpha)$. Using the geometric series, we can write

$$\sum_{i=0}^{\infty} \alpha^i (1-\alpha) k(\alpha) = k(\alpha)$$

for $0 < \alpha < 1$. We call $(1-\alpha)k(\alpha)$ the *equivalent average costs per unit time*. As α tends to 1, from below, the equivalent average costs approach the average costs per unit time:

$$\lim_{\alpha\uparrow 1} (1-\alpha)\left[\frac{\sum_{i=1}^{\infty} \alpha^i c_i p_i}{1-\sum_{i=1}^{\infty} \alpha^i p_i}\right] = \frac{\sum_{i=1}^{\infty} c_i p_i}{\sum_{i=1}^{\infty} i p_i} \qquad (5)$$

using L'Hôpital's rule.

3.4 *Initial cost of investment*

For cost-optimal investment decisions, we are interested in finding an optimum balance between the initial cost of investment and the future cost of maintenance, being the area of life cycle costing. In this situation, the monetary losses over an unbounded horizon are the sum of the initial cost of investment c_0 and the expected discounted future cost:

$$L_\alpha = c_0 + \lim_{n\to\infty} E(K_\alpha(n)).$$

The corresponding expected equivalent average costs per unit time are $(1-\alpha)\, L_\alpha$. For investment decisions, we cannot use the criterion of the expected average costs per unit time,

$$L = \lim_{n\to\infty} \frac{c_0 + E(K(n))}{n} = \lim_{n\to\infty} \frac{E(K(n))}{n},$$

because the contribution of the initial cost to the average costs is ignored.

4 VARIANCE OF COST

The aim of this section is to derive the variance of the discounted costs over an unbounded horizon. This variance can be obtained by applying generating functions. As with the expected value of the discounted costs over an unbounded horizon, we can define an equivalent long-term average variance per unit time. This equivalent average variance per unit time approaches the average variance per unit time, as the discount rate tends to zero from above. The long-term average variance of the costs per unit time is a known result in renewal reward theory [see e.g. Wolff (1989, Chapter 2) and Tijms (1994, Chapter 1)]. As far as the author knows, the expression for the variance of the discounted costs over an unbounded horizon is new.

4.1 *Average variance of the costs per unit time*

The average variance of the costs per unit time can be determined by averaging the variance of the costs over an unbounded horizon. It follows from the first and second moment of the costs over the bounded horizon $(0,n]$, denoted by $E(K(n))$ and $E(K^2(n))$ respectively. These moments solve Eq. (1) as well as the recursive equation

$$\begin{aligned} E(K^2(n)) &= \sum_{i=1}^{n} p_i E([c_i + K(n-i)]^2) = \\ &= \sum_{i=1}^{n} p_i [c_i^2 + 2c_i E(K(n-i)) + E(K^2(n-i))] \end{aligned} \tag{6}$$

for $n = 1,2,3,\ldots$ This equation is obtained by conditioning on the values of the first renewal time T_1. In a slightly different notation, Wolff (1989, Chapter 2) proved that the long-term average variance of the costs per unit term is

$$\begin{aligned} \lim_{n\to\infty} \frac{\mathrm{Var}(K(n))}{n} = &\frac{\mathrm{Var}(c_I)E^2(I) + \mathrm{Var}(I)E^2(c_I)}{[E(I)]^3} \\ &- \frac{2E(I)E(c_I)\mathrm{Cov}(I,c_I)}{[E(I)]^3}. \end{aligned} \tag{7}$$

The random quantity I is the cycle length with cycle cost c_I. If $c_i \equiv 1$ for all $i = 1,2,3,\ldots$ in Eq. (7), then the long-term average variance of the number of renewals per unit time is:

$$\lim_{n\to\infty} \frac{\mathrm{Var}(N(n))}{n} = \frac{\mathrm{Var}(I)}{[E(I)]^3}. \tag{8}$$

This theorem is proved in Feller (1949). He also showed that, as $n \to \infty$, $N(n)$ is asymptotically normal with mean

$$E(N(n)) \sim \frac{n}{E(I)}$$

and variance

$$\mathrm{Var}(N(n)) \sim \frac{n\mathrm{Var}(I)}{[E(I)]^3}.$$

Wolff (1989, Chapter 2) showed that $K(n)$ is asymptotically normal as well.

4.2 *Variance of the discounted costs over an unbounded horizon*

The variance of the expected discounted costs over a bounded time horizon can be obtained by the recursive formulas for the first and second moment of the discounted costs. The former can be found in Eq. (3). The latter can be obtained by conditioning on the values of the first renewal time T_1. The expected value of the square of the discounted costs over the bounded horizon $(0,n]$ can be written as

$$\begin{aligned} E(K_\alpha^2(n)) &= \sum_{i=1}^{n} \alpha^{2i} p_i E([c_i + K_\alpha(n-i)]^2) = \\ &= \sum_{i=1}^{n} \alpha^{2i} p_i [c_i^2 + 2c_i E(K_\alpha(n-i)) + E(K_\alpha^2(n-i))] \end{aligned} \tag{9}$$

for $n = 1,2,3,\ldots$ After some algebra, the second moment of the discounted costs over an unbounded horizon has the form

$$\begin{aligned} &\lim_{n\to\infty} E(K_\alpha^2(n)) = \\ &= 2 \cdot \frac{\sum_{i=1}^{\infty} \alpha^i c_i p_i}{1 - \sum_{i=1}^{\infty} \alpha^i p_i} \frac{\sum_{i=1}^{\infty} \alpha^{2i} c_i p_i}{1 - \sum_{i=1}^{\infty} \alpha^{2i} p_i} + \frac{\sum_{i=1}^{\infty} \alpha^{2i} c_i^2 p_i}{1 - \sum_{i=1}^{\infty} \alpha^{2i} p_i} = \\ &= 2 \cdot \frac{E(\alpha^I c_I)}{1 - E(\alpha^I)} \frac{E(\alpha^{2I} c_I)}{1 - E(\alpha^{2I})} + \frac{E(\alpha^{2I} c_I^2)}{1 - E(\alpha^{2I})}. \end{aligned} \tag{10}$$

The proof can be obtained by rewriting the recursive relation (9) in terms of the corresponding generating functions. The proof will be presented in a future paper.

The variance of the discounted costs over an unbounded horizon can now be easily derived by combining Eqs. (4) and (10); that is,

$$\lim_{n\to\infty} \mathrm{Var}(K_\alpha(n)) = \lim_{n\to\infty} [E(K_\alpha^2(n)) - E^2(K_\alpha(n))]. \tag{11}$$

An interesting special case arises if $c_i \equiv 1$ for all $i = 1,2,3,\ldots$ Then, the variance of the number of imaginary "discounted renewals" over an unbounded horizon is

$$\lim_{n\to\infty} \mathrm{Var}(N_\alpha(n)) = \frac{[\sum_{i=1}^{\infty} \alpha^{2i} p_i] - (\sum_{i=1}^{\infty} \alpha^i p_i)^2}{[1 - \sum_{i=1}^{\infty} \alpha^{2i} p_i](1 - \sum_{i=1}^{\infty} \alpha^i p_i)^2}.$$

4.3 *Equivalent average variance of the costs per unit time*

As with the expected value of the costs, the equivalent average variance of the costs per unit time relate to the average variance per unit time and the variance of the discounted costs over an unbounded horizon. To establish this relation, let us define the *equivalent average variance of the costs per unit time* to be

$$(1-\alpha^2)\lim_{n\to\infty}\mathrm{Var}(K_\alpha(n)).$$

As α tends to 1, from below, the equivalent average variance of the costs per unit time approaches the average variance of the costs per unit time; that is,

$$\lim_{\alpha\uparrow 1}(1-\alpha^2)\lim_{n\to\infty}\mathrm{Var}(K_\alpha(n)) = \lim_{n\to\infty}\frac{\mathrm{Var}(K(n))}{n}. \quad (12)$$

The proof is rather tedious and will be presented in a future paper. If $c_i \equiv 1$ for all $i = 1,2,3,\ldots$, then Eq. (12) collapses to

$$\lim_{\alpha\uparrow 1}(1-\alpha^2)\lim_{n\to\infty}\mathrm{Var}(N_\alpha(n)) = \lim_{n\to\infty}\frac{\mathrm{Var}(N(n))}{n}.$$

In Section 4, it was mentioned that the non-discounted costs over an unbounded horizon are asymptotically normal. Therefore, it is to be expected that the discounted costs over an unbounded horizon are approximately asymptotically normal, as long as the discount rate is close to zero.

5 CONTINUOUS-TIME PROCESSES

Up to now, we have studied only discrete-time renewal processes. Similar results can be obtained for continuous-time renewal processes; basically by replacing 'summations' with 'integrals'. The variance of the discounted costs over an unbounded horizon should now be computed by applying Laplace transforms instead of generating functions.

Let $F(t)$ be the cumulative probability distribution of the continuous renewal time $T \geq 0$ and let $c(t)$ be the cost associated with a renewal at time t. Using a terminating renewal argument and applying Laplace transforms, the expected discounted cost over an unbounded horizon can then be written as

$$\lim_{t\to\infty}E(K_\alpha(t)) = \frac{\int_0^\infty \alpha^t c(t)dF(t)}{1-\int_0^\infty \alpha^t dF(t)}, \quad (13)$$

where $K_\alpha(t)$ represents the expected discounted costs in the bounded time-interval $(0,t]$, $t > 0$. Eq. (13) generalises the work of Berg (1980) and Fox (1966), who studied age and block replacement policies with discounting. Rackwitz (2000) studied the situation $c(t) \equiv c > 0$ for all $t \geq 0$. In a similar manner, the second moment of the discounted costs over an unbounded horizon is given by

$$\lim_{t\to\infty}E(K_\alpha^2(t)) =$$

$$= 2\cdot\frac{\int_0^\infty \alpha^t c(t)dF(t)}{1-\int_0^\infty \alpha^t dF(t)}\frac{\int_0^\infty \alpha^{2t} c(t)dF(t)}{1-\int_0^\infty \alpha^{2t} dF(t)} + \frac{\int_0^\infty \alpha^{2t} c^2(t)dF(t)}{1-\int_0^\infty \alpha^{2t} dF(t)}.$$

The continuous-time versions of the discrete-time limit theorems (5) and (12) are as follows. As α tends to 1, from below, the equivalent average costs per unit time approach the average costs per unit time:

$$\lim_{\alpha\uparrow 1}(-\log\alpha)\lim_{t\to\infty}E(K_\alpha(t)) = \lim_{t\to\infty}\frac{E(K(t))}{t} \quad (14)$$

using L'Hôpital's rule. Note that

$$\int_0^\infty \alpha^t dt = -\frac{1}{\log\alpha}.$$

As α tends to 1, from below, the equivalent average variance of the costs per unit time approaches the average variance of the costs per unit time; that is,

$$\lim_{\alpha\uparrow 1}(-\log\alpha^2)\lim_{t\to\infty}\mathrm{Var}(K_\alpha(t)) = \lim_{t\to\infty}\frac{\mathrm{Var}(K(t))}{t}. \quad (15)$$

It should be noted that the limits for t approaching infinity on the right-hand side of Eqs. (14) and (15) do not always exist. In order for these limits to exist, the renewal times should have a so-called non-lattice distribution. A random quantity X, and its distribution, are called *lattice* if for some $d > 0$,

$$\textstyle\sum_{i=1}^\infty \Pr\{X = id\} = 1.$$

For a detailed discussion on the existence of pointwise limits of functions arising in renewal theory, see Wolff (1989, Chapter 2).

6 ILLUSTRATIONS

This section describes three examples in which the formulas of the expected value and the variance of the discounted costs over an unbounded horizon can be applied.

6.1 *Flood prevention*

Let the discrete inter-occurrence times of a flood be distributed as a geometric distribution with parameter p, so that

$$p_i = p(1-p)^{i-1}, \quad i = 1,2,3,\ldots$$

The parameter p can be interpreted as the probability of occurrence of the flood per unit time. Furthermore, assume the cost of flood damage to be $c > 0$.

The expected value and the variance of the discounted costs over an unbounded horizon are

$$\lim_{n\to\infty} E(K_\alpha(n)) = \frac{\alpha}{1-\alpha}\cdot pc$$

and

$$\lim_{n\to\infty} \mathrm{Var}(K_\alpha(n)) = \frac{\alpha^2}{1-\alpha^2}\cdot p(1-p)c^2,$$

respectively. When the building cost is included as well, these formulas can be used to design flood defences (see e.g. Van Dantzig, 1956).

6.2 *Poissonian failure process*

The continuous-time analogue of the discrete geometric distribution is the exponential distribution. Let failures occur according to a Poisson process with arrival rate λ, then the inter-occurrence failure time is exponentially distributed with mean λ^{-1}. With the failure cost being $c > 0$, the expected discounted costs over an unbounded horizon are

$$\lim_{t\to\infty} E(K_\alpha(t)) = -\frac{1}{\log\alpha}\cdot\lambda c.$$

Accordingly, the variance of the discounted costs over an unbounded horizon is

$$\lim_{t\to\infty} \mathrm{Var}(K_\alpha(t)) = -\frac{1}{\log\alpha^2}\cdot\lambda c^2.$$

In Kuschel & Rackwitz (2000) and Rackwitz (2000), the parameter λ represents the outcrossing rate of a Poissonian failure process.

6.3 *Age replacement*

A well-known preventive maintenance strategy is the age replacement strategy. Under an age replacement policy, a replacement is carried out at age k (preventive replacement) *or* at failure (corrective replacement), whichever occurs first, where $k = 1,2,3,\ldots$ A preventive replacement entails a cost c_P, whereas a corrective replacement entails a cost c_F, where $0 < c_P \le c_F$.

As a simplified example, we study the maintenance of a cylinder on a swing bridge (adapted from Van Noortwijk, 1998). Preventive maintenance of a cylinder mainly consists of replacing the guide bushes and plunger, and replacing the packing of the piston rod. In the event of corrective maintenance, the cylinder has to be replaced completely because too much damage has occurred. The cost of preventive maintenance c_P is 30,000 Dutch guilders, whereas the cost of corrective maintenance c_F is 100,000 Dutch guilders. Both maintenance actions bring the cylinder back into its “good as new state”. The rate of deterioration is based on periodic life-time-extending maintenance, in terms of cleaning and sealing the cylinder. The time at which the expected condition equals the failure level is 15 years. On the basis of the stochastic deterioration process described by a gamma process, the annual probabilities of failure p_i, $i = 1,2,3,\ldots$, can be easily computed. It follows from Eq. (4) that the expected discounted costs of age replacement over an unbounded horizon are

$$\lim_{n\to\infty} E(K_\alpha(n)) = \frac{(\sum_{i=1}^k \alpha^i p_i)c_F + \alpha^k(1-\sum_{i=1}^k p_i)c_P}{1-[(\sum_{i=1}^k \alpha^i p_i)+\alpha^k(1-\sum_{i=1}^k p_i)]}.$$

Similarly, the second moment of the discounted costs over an unbounded horizon can be written as

$$\begin{aligned}
&\lim_{n\to\infty} E(K_\alpha^2(n)) = \\
&= 2\cdot\frac{(\sum_{i=1}^k \alpha^i p_i)c_F + \alpha^k(1-\sum_{i=1}^k p_i)c_P}{1-[(\sum_{i=1}^k \alpha^i p_i)+\alpha^k(1-\sum_{i=1}^k p_i)]}\times \\
&\quad\times\frac{(\sum_{i=1}^k \alpha^{2i} p_i)c_F + \alpha^{2k}(1-\sum_{i=1}^k p_i)c_P}{1-[(\sum_{i=1}^k \alpha^{2i} p_i)+\alpha^{2k}(1-\sum_{i=1}^k p_i)]} + \\
&\quad+\frac{(\sum_{i=1}^k \alpha^{2i} p_i)c_F^2 + \alpha^{2k}(1-\sum_{i=1}^k p_i)c_P^2}{1-[(\sum_{i=1}^k \alpha^{2i} p_i)+\alpha^{2k}(1-\sum_{i=1}^k p_i)]}.
\end{aligned}$$

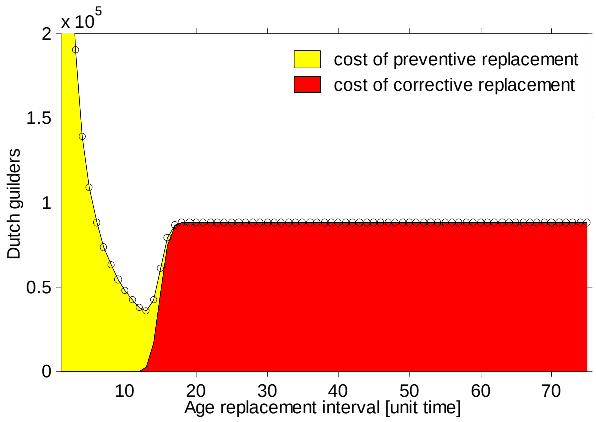

Figure 1. The expected value of the discounted costs over an unbounded horizon as a function of the age replacement interval k, $k = 1,\ldots,75$.

The optimal age replacement interval is the interval for which the expected discounted costs over an unbounded horizon are minimal. On the basis of an annual discount rate of 5%, the expected discounted costs of preventive and corrective maintenance are displayed in Figure 1. The expected discounted costs over an unbounded horizon are minimal for an age replacement interval of 13 years. The standard deviation of the discounted costs over an unbounded horizon is shown in Figure 2. For an age replacement interval of 15 years, the standard deviation is at a maximum.

Note that the replacement model can also be applied for determining the initial resistance of a structure, which optimally balances the initial cost of investment c_P against the future cost of maintenance.

7 CONCLUDING REMARKS

The paper presents explicit formulas for both the expected value and the variance of the discounted costs over an unbounded horizon. It is shown that there is an interesting connection between the expressions for the discounted costs and well-known results of renewal reward theory (with respect to the long-term average costs per unit time). The variance of the discounted costs over an unbounded horizon is useful to determine (approximate) uncertainty bounds.

The formulas can be applied in situations where regenerative cycles can be identified; that is, after each renewal we start (in a statistical sense) all over again. The advantage of the expressions is that they can be easily computed. Even when we have to rely on Monte Carlo simulation for calculating the probabilistic characteristics of both the renewal cycle length and the renewal cycle cost, the two expressions can be easily used. The reason is that both expressions can be reformulated solely in terms of expected values of simple functions of the renewal time. Although renewal times are mainly influenced by failures, they may also depend on inspections, repairs, replacements, and lifetime extensions.

Due to the discount rate, the expressions for the expected value and the variance of the discounted costs over an unbounded horizon may serve as a good approximation in situations with a bounded time horizon larger than fifty years. On the other hand, this paper also presents recursive formulas that can be used to calculate the expected discounted costs over a bounded horizon.

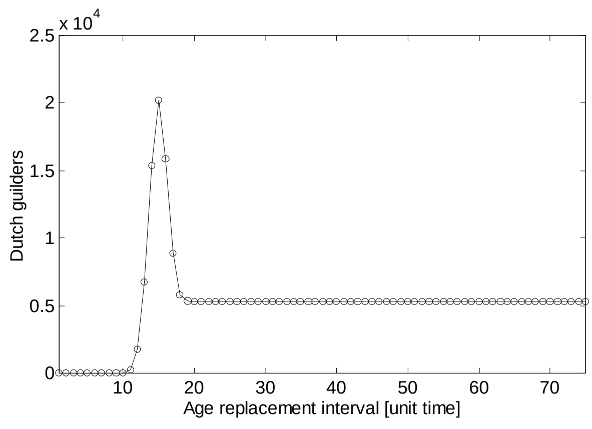

Figure 2. The standard deviation of the discounted costs over an unbounded horizon as a function of the age replacement interval k, $k = 1,\ldots,75$.

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