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THE USE OF LIFETIME DISTRIBUTIONS IN BRIDGE REPLACEMENT MODELLING

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Abstract. *This paper proposes a new method to determine lifetime distributions for concrete bridges and to compute the expected cost of replacing a bridge stock. The uncertainty in the lifetime of a bridge can best be represented with a Weibull distribution. It is recommended to fit this Weibull distribution on the basis of aggregating the lifetimes of demolished bridges (complete observations) and the ages of current bridges (right-censored observations). Using renewal theory, the future expected cost of replacing the bridge stock can then be easily determined while taking account of the current bridge ages and the corresponding uncertainties in the future replacement times. The proposed methodology is used to estimate the cost of replacing the Dutch stock of concrete bridges as a function of time.*

1 INTRODUCTION

The Dutch Directorate General for Public Works and Water Management is responsible for the management of road bridges in the Netherlands (Klatter et al. [5]). This management can be optimised by balancing maintenance and replacement of bridges using life-cycle costing. To calculate the life-cycle cost, information on the time and cost of bridge replacement is needed. Therefore, this paper has two objectives: determining lifetime distributions for concrete bridges and computing the expected cost of replacement of the current bridge stock as a function of time and age. This paper mainly focusses on bridge replacement (defined as 'essential maintenance' in the BRIME report [7]).

The first objective is to perform a statistical analysis on lifetimes of demolished bridges and ages of existing bridges. However, the general opinion on estimating a lifetime solely on the basis of bridge replacement times is that the resulting expected lifetime is often considerably underestimated. Although this statement is confirmed by our results, we claim that a statistical analysis is nevertheless useful. The underestimation problem can be resolved by fitting a probability distribution to both the lifetimes of demolished bridges (complete observations) and the current ages of existing bridges (right-censored observations). The so-obtained estimates of the expected lifetime of a concrete bridge are more in accordance with the usual design life. In order to properly model the ageing of bridges, the use of the Weibull distribution is recommended. Using the maximum-likelihood method, a Weibull distribution can be fitted to both complete and right-censored observations. An advantage of the Weibull distribution is that the conditional probability distribution of the residual lifetime given the current age can be analytically expressed.

The second objective is to estimate the future expected cost of replacing the bridge stock as a function of time. This can be done by applying techniques from renewal theory. As a matter of fact, this paper contains all the mathematical formulas that are needed to explicitly compute the expected cost of bridge replacement over a bounded horizon when the current ages are given.

The outline of this paper is as follows. The statistical methodology to determine the lifetime distribution of concrete bridges on the basis of lifetimes of demolished bridges and ages of the current bridge stock is presented in Section 2. Mathematical formulas for the expected value of the future cost of bridge replacement over a bounded horizon are derived in Section 3. Section 4 describes a case study on estimating the lifetime distribution of the Dutch concrete bridges and the expected future cost of replacement of the bridge stock. Conclusions are given in Section 5.

2 LIFETIME DISTRIBUTION OF BRIDGES

A probability distribution which is especially useful for modelling ageing is the Weibull distribution (Barlow & Proschan [1]). A random variable X has a Weibull distribution with shape parameter $a > 0$ and scale parameter $b > 0$ if the probability density function of X is given by

$$\ell\,(x|a,b) = \mathrm{We}(x|a,b) = \frac{a}{b}\left[\frac{x}{b}\right]^{a-1} \exp\left\{-\left[\frac{x}{b}\right]^{a}\right\} I_{(0,\infty)}(x). \tag{1}$$

The survival function is defined by

$$\bar{F}(x|a,b) = 1 - F(x|a,b) = \exp\left\{-\left[\frac{x}{b}\right]^{a}\right\} \tag{2}$$

with expected value $E(X) = b\Gamma(a^{-1}+1)$.

For calculating the future replacement cost of the current bridge stock, we have to account for their ages. To achieve this, we condition on the current life or age y and determine the conditional probability that the lifetime X exceeds x given $X > y$; that is,

$$\Pr\{X > x|X > y\} = \bar{F}(x|a,b,y) = \exp\left\{-\left[\frac{x}{b}\right]^{a} + \left[\frac{y}{b}\right]^{a}\right\} \tag{3}$$

for $x > y$. The corresponding probability density function of this so-called left-truncated Weibull distribution is then given by

$$\mathrm{LTW}(x|a,b) = \frac{a}{b}\left[\frac{x}{b}\right]^{a-1} \exp\left\{-\left[\frac{x}{b}\right]^{a} + \left[\frac{y}{b}\right]^{a}\right\} I_{(y,\infty)}(x). \tag{4}$$

This density represents the uncertainty in the lifetime of a bridge having current age y, where the residual (or excess) lifetime is defined as $X - y$. The statistical properties of the left-truncated Weibull distribution are derived by Wingo [6].

For estimating the parameters of the Weibull distribution for bridge lifetimes, there are two types of observations available: complete lifetimes of demolished bridges and right-censored lifetimes of existing bridges. Although the latter observations do not contain actual lifetimes, they are nevertheless a valuable source of information. At least we know that the lifetimes of the existing bridges will be larger than their current ages. Using maximum-likelihood estimation, the complete and right-censored observations can be used to estimate the parameters a and b of the Weibull distribution. Let $\mathbf{x} = (x_1, \ldots, x_r)'$ denote a random sample of r complete lifetimes and $\mathbf{y} = (y_1, \ldots, y_m)'$ a random sample of m right-censored lifetimes. Using Eqs. (1) and (2), the corresponding likelihood function can be written as

$$\ell\,(\mathbf{x},\mathbf{y}|\,a,b) = \textstyle\prod_{i=1}^{r} \ell\,(x_i|\,a,b) \prod_{j=1}^{m} \bar{F}\,(y_j|\,a,b)\,. \tag{5}$$

The maximum-likelihood estimators $\hat{a}$ and $\hat{b}$ can be computed by numerically maximising the logarithm of the likelihood function (5).

3 EXPECTED COST OF REPLACEMENT

The purpose of this section is to derive the expected cost of bridge replacement while taking account of the uncertainty in the lifetime. We assume that bridge replacement can be approximately modelled as a discrete renewal process, whereby the renewals are the replacements. After each renewal we start (in a statistical sense) all over again. A discrete renewal process $\{N(n), n = 1, 2, 3, \ldots\}$ is a non-negative integer-valued stochastic process that registers the successive renewals in the time interval $(0, n]$. Let the renewal times $T_1, T_2, T_3, \ldots,$ be non-negative, independent, identically distributed, random quantities having the discrete probability function

$$\Pr\{T_k = i\} = p_i = F(i|a, b, 0) - F(i - 1|a, b, 0), \tag{6}$$

$i = 1, 2, \ldots$, where p_i represents the probability of a renewal in unit time i. This probability function is a discretised Weibull distribution.

It is assumed that the cost associated with a renewal does not change over time and is equal to c. The expected cost over the bounded horizon $(0, n]$, denoted by $E(K(n))$, follows then directly from the expected number of renewals $E(N(n))$: $E(K(n)) = cE(N(n))$. According to Van Noortwijk & Peerbolte [4], the expected number of renewals is a solution of the recursive equation

$$E(N(n)) = \sum_{i=1}^{n} p_i \left[1 + E(N(n - i))\right] \tag{7}$$

for $n = 1, 2, 3 \ldots$ and $N(0) \equiv 0$. To obtain this equation, we condition on the values of the first renewal time T_1 and apply the law of total probability. With the occurrence of the event $T_1 = i$, the number of renewals is one plus the additional expected number of renewals during the interval $(i, n]$, $i = 1, \ldots, n$. Using the discrete renewal theorem (see Feller [2, Ch. 12 & 13] and Karlin & Taylor [3, Ch. 3]), the expected long-term average number of renewals per unit time is

$$\lim_{n \to \infty} \frac{E(N(n))}{n} = \frac{1}{\sum_{i=1}^{\infty} i p_i} = \frac{1}{\mu} \tag{8}$$

being the reciprocal of the mean lifetime μ. As $n \to \infty$, the expected long-term average cost per unit time approaches c/μ.

The expression for the expected number of renewals over a bounded horizon (7) can be extended to the situation in which the first bridge has age $y \geq 0$. For this purpose, the probability distribution of the residual lifetime can be discretised in terms of

$$\Pr\{\tilde{T} = i|y\} = q_i(y) = F(y + i|a, b, y) - F(y + i - 1|a, b, y), \quad i = 1, 2, \ldots \tag{9}$$

The expected number of renewals in time interval $(0, n]$ when the first bridge has age y can then be written as

$$E(\tilde{N}(n, y)) = \sum_{i=1}^{n} q_i(y) \left[1 + E(N(n - i))\right], \tag{10}$$

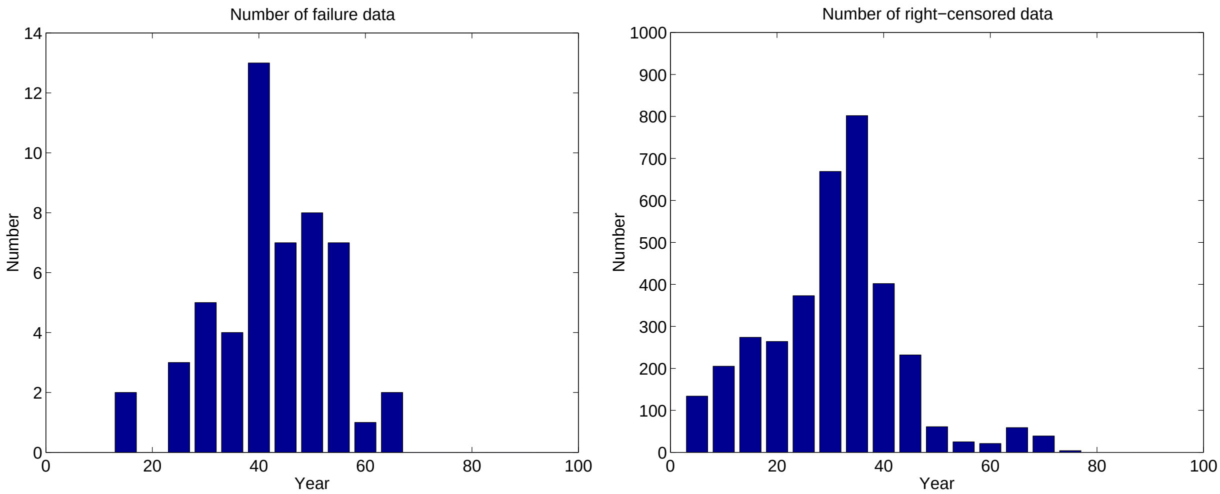

Figure 1: Histogram of (a) complete lifetimes and (b) right-censored lifetimes (current ages).

where $E(N(n))$ is given by Eq. (7). Because the renewal process only starts from the second renewal on, the non-negative integer-valued stochastic process $\{\tilde{N}(n,y), n = 1,2,3,\ldots\}$ is called a delayed renewal process (see Karlin & Taylor [3, Ch. 5]).

In general, the expected cost of replacement of a bridge stock can now be obtained by summing the expected replacement cost over the current ages $y_1,\ldots,y_m$:

$$E(\tilde{K}(n)) = \textstyle\sum_{j=1}^{m} c_j E(\tilde{N}(n,y_j)), \tag{11}$$

where c_j is the cost of replacing the jth concrete bridge. Accordingly, the expected cost of bridge replacement in unit time i is simply $E(\tilde{K}(i)) - E(\tilde{K}(i-1))$, $i = 1,\ldots,n$.

4 DUTCH STOCK OF CONCRETE BRIDGES

The proposed method for estimating lifetime distributions and future replacement costs has been applied to the Dutch stock of concrete bridges in the main road network. This stock consists of concrete underpasses, viaducts and bridges. The main purpose of this paper is to present the methodology rather than to perform a thorough comparative statistical analysis for all the different categories of structures. Furthermore, the data that has been used for determining the lifetime distribution and the expected replacement cost must still be improved. At the time of writing, the data on lifetimes of demolished bridges and ages of current bridges was not yet complete.

4.1 Estimation of lifetime distribution

The observed lifetimes and ages of concrete bridges were aggregated. In doing so, we gathered $r = 52$ lifetimes of demolished bridges (the complete observations in Figure 1a) and $m = 3564$ ages of existing bridges (the right-censored observations in Figure 1b). Because the right-censored observations were only available in terms of units of time of

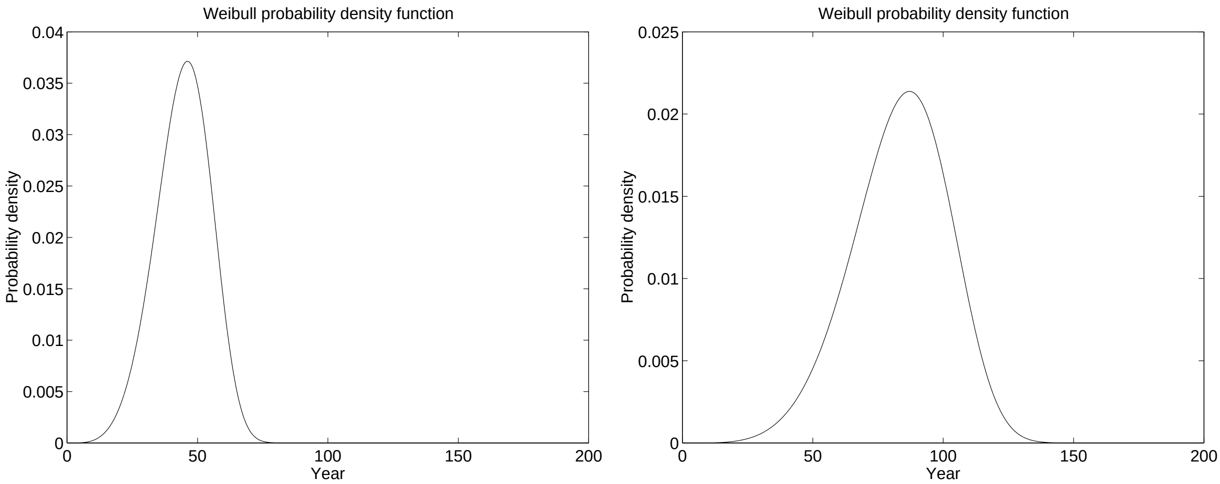

Figure 2: Weibull distribution estimated on the basis of (a) complete lifetimes and (b) both complete and right-censored lifetimes.

five years, bridge replacement was modelled as a discrete-time renewal process. For 150 concrete bridges, the year of construction was unknown leading to a total number of concrete bridges in the Netherlands of $\tilde{m} = 3714$ (adapted from Klatter et al. [5]).

A statistical analysis has been performed for complete lifetimes only as well as for the combination of complete and right-censored lifetimes. The general opinion of bridge maintenance managers is that a statistical analysis of replaced bridges is not useful, because the fitted lifetime distribution would underestimate the expected lifetime considerably (about 40 to 50 years instead of the usual design life of 80 to 100 years). The main reason for this is that most demolished bridges were *not* replaced due to technical failure, but due to a change in functional or economical requirements. Examples are bridges replaced because of insufficient load-carrying capacity due to an unexpected increase of heavy traffic intensity. Unfortunately, there was not enough information available for making a distinction between the technical, functional and economical lifetime. Therefore, the observed lifetimes of demolished bridges can be either of these three and they were analysed as a whole. Furthermore, it should be noted that possible changes in bridge design over time could not yet be taking into account. This will be investigated in the future.

As expected, our statistical analysis results in an underestimation of the expected lifetime: the mean is 44 years with a coefficient of variation of 0.24. The corresponding maximum-likelihood estimators of the shape parameter a and the scale parameter b of the Weibull distribution are $\hat{a} = 4.2$ and $\hat{b} = 48.5$, respectively. The resulting Weibull probability density function based on complete observations is shown in Figure 2a. However, when the current ages of the concrete bridge stock are included, the results change considerably. The expected lifetime increases from 44 to 83 years! The coefficient of variation does not change much; its value is 0.22. The maximum-likelihood estimates are $\hat{a} = 5.2$ and $\hat{b} = 90.2$. The Weibull density function based on both complete and

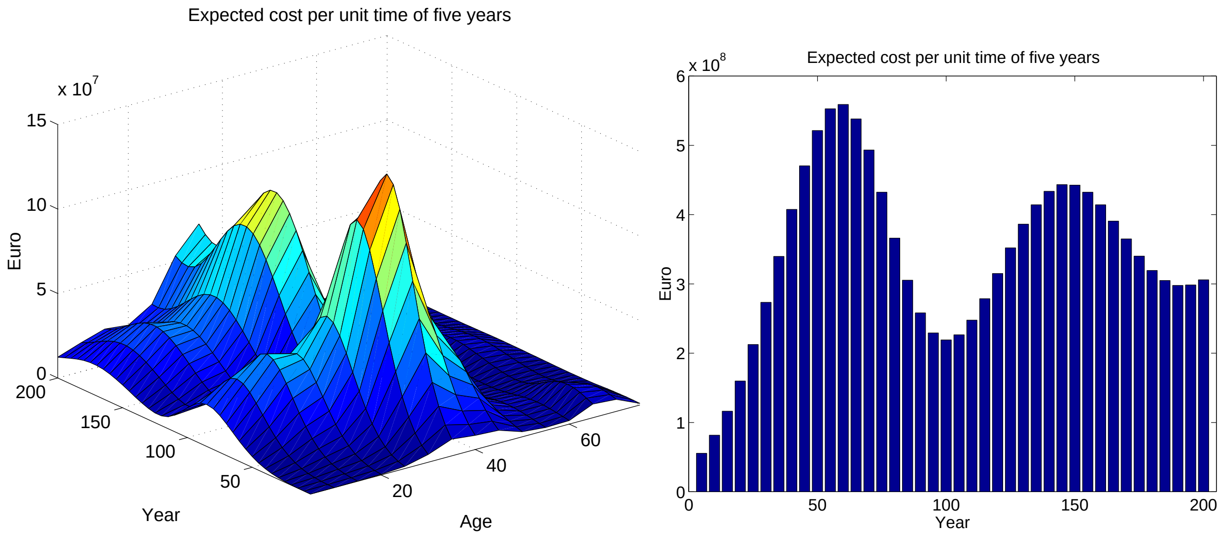

Figure 3: Expected cost per unit time i, $i = 1, \ldots, n$: (a) as a function of age or (b) summed over all ages.

right-censored observations is shown in Figure 2b. Because the Weibull shape parameter is larger than unity, both situations clearly represent ageing (Barlow & Proschan [1]).

4.2 Calculation of replacement cost

When the lifetime distribution is based on both complete and right-censored observations, the expected replacement cost over the bounded horizon $(0, n]$ can be computed with Eq. (11). For the purpose of illustration, we assume the replacement cost of a concrete bridge to be the same for all bridges. Although an old bridge is seldom replaced by the same type of bridge, it is difficult to accurately assess the cost of such a new bridge. Let the cost of replacement therefore be independent of time. From Klatter et al. [5], the cost of replacement of one bridge directly follows from the replacement value of the complete stock of $\tilde{m} = 3714$ bridges being 6.4×10^9 Euro: that is, $c = 1.7$ million Euro.

In Figure 3a, the expected cost per unit time of five years is shown as a function of the unit time and the bridge's age. Summing over all the concrete bridges and their corresponding ages gives the expected cost per unit time as shown in Figure 3b. As expected, the uncertainty in the second replacement time is larger than the uncertainty in the first replacement time. As the time horizon approaches infinity, the expected long-term average cost per unit time of five years approaches $m \cdot c/\mu = 357$ million Euro (compare with the limit (8)). This average cost is thus 71 million Euro per year. Indefinitely far in the future, our delayed renewal process thus becomes a stationary renewal process for which the expected cost per unit time finds an equilibrium value. To account for the replacement cost of the 150 concrete bridges with unknown years of construction, the expected replacement cost per unit time shown in Figure 3 should finally be multiplied by a factor $\tilde{m}/m = 1.04$. The cost of preventive and lifetime-extending (routine) maintenance is not included in these figures; only the cost of essential maintenance is considered.

5 CONCLUSIONS

A statistical analysis for determining the lifetime distribution of concrete bridges in the Netherlands has been presented. A Weibull distribution was fitted to both complete lifetimes of demolished bridges and current ages of existing bridges. Unlike the average value of the observed complete lifetimes, the expected value of the Weibull lifetime distribution was in agreement with the usual design life. Advantages of representing the uncertainty in the lifetime of bridges with a Weibull distribution are the possibility to properly model ageing and to analytically derive the conditional probability density function of the residual lifetime when the current age is given.

Using the discrete renewal theorem, the so-obtained Weibull distribution has been used to determine the future expected cost of replacing the bridge stock. In calculating this cost, the ages of the individual bridges were taken into account. In a case study, the expected future cost of replacement of the Dutch concrete bridges has been estimated. Taking account of the uncertainties in the replacement times has the advantage that the cost is more spread out over time than in the deterministic case. Since the methodology appears to work well, it could also be applied to other (sub)categories of structures. The proposed methodology can be extended by distinguishing different types of lifetime (such as technical, functional and economical), considering a possible change of bridge design over time and varying the replacement cost over the individual structures.

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