Gamma processes for time-dependent reliability of structures

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ABSTRACT: In structural engineering, a distinction can be made between a structure's resistance and its stress. A failure may then be defined as the event in which—due to deterioration—the resistance drops below the stress. This paper proposes a method to combine the two stochastic processes of deteriorating resistance and fluctuating load for computing the time-dependent reliability of a structural system. As deterioration is often uncertain and non-negative, it can best be regarded as a gamma process. A gamma process is a stochastic process with independent non-negative increments having a gamma distribution with identical scale parameter. The stress is assumed to be a stochastic process of random loads exceeding a certain threshold and occurring according to a Poisson process. The variability of the random loads is modelled by a peaks-over-threshold distribution (such as the generalised Pareto distribution). These stochastic processes of deterioration and load can be combined in a straightforward manner to perform a time-dependent reliability analysis.

1 INTRODUCTION

In structural engineering, a distinction can often be made between a structure's resistance and its applied stress. A failure may then be defined as the event in which—due to deterioration—the resistance drops below the stress. This paper proposes a method to combine the two stochastic processes of deteriorating resistance and fluctuating load for computing the time-dependent reliability of a structural system.

As deterioration is generally uncertain and non-decreasing, it can best be regarded as a gamma process (see, e.g., Abdel-Hameed, 1975). In words, a gamma process is a stochastic process with independent non-negative increments having a gamma distribution with identical scale parameter. The stress is assumed to be a stochastic process of random loads occurring at random times in the same way as it was done by Ellingwood and Mori (1993). We adopt their assumption of the loads occurring according to a Poisson process. The difference is that we consider only loads larger than a certain threshold rather than the loads themselves, because the Poisson assumption is then better justified. In order to determine the probability distribution of extreme loads (such as water levels, discharges, waves, winds, temperatures and rainfall), we use the peaks-over-threshold method. Exceedances over the threshold are typically assumed to have a generalised Pareto distribution.

Given the above definitions of the stochastic processes of deterioration and load, they can now be combined in a straightforward manner to compute the time-dependent reliability. This method extends the approach of Mori and Ellingwood (1994), who modelled the uncertainty in the deterioration process by posing a probability distribution on the damage growth rate rather than assuming a stochastic process with independent increments. In order to combine the extreme load events with the deteriorating resistance, we assume the load threshold exceedances as a series of pulses having an "infinitely small" duration. As a consequence, we may then suppose that the resistance does not degrade during extreme load events and that failure only occurs during exceedances of the load threshold.

The paper is organised as follows. The stochastic processes of deteriorating resistance and fluctuating load are presented in Section 2 and 3, respectively. These two stochastic processes are combined for the purpose of time-dependent reliability analysis in Section 4. The proposed method is applied to a dike reliability problem in Section 5 and conclusions are formulated in Section 6.

2 STOCHASTIC DETERIORATION PROCESS

In structural engineering, a distinction can often be made between a structure's resistance (e.g. the crest-level of a dike) and its applied stress (e.g. the water level to be withstood). A failure may then be defined as the event in which the deteriorating resistance drops below the stress.

Because deterioration is uncertain, it can best be regarded as a time-dependent stochastic process $\{X(t),\, t \geq 0\}$ where $X(t)$ is a random quantity for all $t \geq 0$. At first glance, it seems possible to represent the uncertainty in a deterioration process by the normal distribution. This probability distribution has been used for modelling the exchange value of shares and the movement of small particles in fluids and air. A characteristic feature of this model, also denoted by the Brownian motion with drift (see, e.g., Karlin and Taylor, 1975, Chapter 7), is that a structure's resistance alternately increases and decreases, like the exchange value of a share. For this reason, the Brownian motion is inadequate in modelling deterioration which proceeds in one direction. For example, a dike of which the height is subject to a Brownian deterioration can, according to the model, spontaneously rise up, which of course cannot occur in practice.

In order for the stochastic deterioration process to proceed in one direction, we can best consider it as a gamma process (see, e.g., van Noortwijk et al., 1997). In words, a gamma process is a stochastic process with independent non-negative increments (e.g. the increments of crest-level decline of a dike) having a gamma distribution with identical scale parameter. In the case of a gamma deterioration, dikes can only sink. To the best of the authors' knowledge, Abdel-Hameed (1975) was the first to propose the gamma process as a proper model for deterioration occurring random in time. The gamma process is suitable to model gradual damage monotonically accumulating over time, such as wear, fatigue, corrosion, crack growth, erosion, consumption, creep, swell, etc. For the mathematical aspects of gamma processes, see Dufresne et al. (1991), Ferguson and Klass (1972), Singpurwalla (1997), and van der Weide (1997). An advantage of gamma processes is that the required calculations are relatively straightforward.

In mathematical terms, the gamma process is defined as follows. Recall that a random quantity X has a gamma distribution with shape parameter $v > 0$ and scale parameter $u > 0$ if its probability density function is given by:

$$\text{Ga}(x|\, v, u) = \frac{u^v}{\Gamma(v)}\, x^{v-1} \exp\{-ux\}\, I_{(0,\infty)}(x),$$

where $I_A(x) = 1$ for $x \in A$ and $I_A(x) = 0$ for $x \notin A$. Furthermore, let $v(t)$ be a non-decreasing, right continuous, real-valued function for $t \geq 0$ with $v(0) \equiv 0$. The gamma process with shape function $v(t) > 0$ and scale parameter $u > 0$ is a continuous-time stochastic process $\{X(t), t \geq 0\}$ with the following properties:

1. $X(0) = 0$ with probability one;
2. $X(\tau) - X(t) \sim \text{Ga}(v(\tau) - v(t), u)$ for $\tau > t \geq 0$;
3. $X(t)$ has independent increments.

Let $X(t)$ denote the deterioration at time t, $t \geq 0$, and let the probability density function of $X(t)$, in conformity with the definition of the gamma process, be given by

$$f_{X(t)}(x) = \text{Ga}\,(x\,|\,v(t), u) \tag{1}$$

with

$$E(X(t)) = \frac{v(t)}{u}, \quad \text{Var}(X(t)) = \frac{v(t)}{u^2}. \tag{2}$$

A component is said to fail when its deteriorating resistance, denoted by $R(t) = r_0 - X(t)$, drops below the stress s (see Figure 1). For the time being, we assume both the initial resistance r_0 and the stress s to be deterministic. Let the time at which failure occurs be denoted by the lifetime T. Due to the gamma distributed deterioration, Eq. (1), the lifetime distribution can then be written as:

$$F(t) = \Pr\{T \leq t\} = \Pr\{X(t) \geq r_0 - s\} = \\ = \int_{x=r_0-s}^{\infty} f_{X(t)}(x)\, dx = \frac{\Gamma(v(t), [r_0 - s]u)}{\Gamma(v(t))},$$

where $\Gamma(a, x) = \int_{t=x}^{\infty} t^{a-1} e^{-t}\, dt$ is the incomplete gamma function for $x \geq 0$ and $a > 0$. Using the chain rule for differentiation, the probability density function of the lifetime is

$$f(t) = F'(t) = \frac{\partial}{\partial \tilde{v}} \left[\frac{\Gamma(\tilde{v}, [r_0 - s]u)}{\Gamma(\tilde{v})} \right] \Bigg|_{\tilde{v}=v(t)} v'(t) \tag{3}$$

under the assumption that the shape function $v(t)$ is differentiable. The partial derivative in Eq. (3) can be calculated by the algorithm of Moore (1982). Using a series expansion and a continued fraction expansion, this algorithm computes the first and second partial derivatives with respect to x and a of the incomplete gamma integral

$$P(a, x) = \frac{1}{\Gamma(a)} \int_{t=0}^{x} t^{a-1} e^{-t}\, dt = \frac{\Gamma(a) - \Gamma(a, x)}{\Gamma(a)}.$$

Under the assumption of modelling temporal variability in the deterioration in terms of a gamma process, the question which remains to be answered is how its expected deterioration increases over time.

Empirical studies show that the expected deterioration at time t is often proportional to a power law:

$$E(X(t)) = \frac{v(t)}{u} = \frac{at^b}{u} \tag{4}$$

for some physical constants $u > 0$, $a > 0$, and $b > 0$. Some examples are the expected degradation of concrete due to corrosion of reinforcement (linear: $b = 1$), sulfate attack (parabolic: $b = 2$), and diffusion-controlled aging (square root: $b = 0.5$) studied by Ellingwood and Mori (1993). Because there is often engineering knowledge available about the shape of the expected deterioration (for example, in terms of the parameter b in Eq. (4)), this parameter may be assumed constant. In the event of expected deterioration in terms of a power law, the parameters a and u yet must be assessed by using expert judgment and/or statistics. It should be noted that the analysis of this paper is not restricted to using a power law for modelling the expected deterioration over time. As a matter of fact, any shape function $v(t)$ suffices, as long as it is a non-decreasing, right continuous, and real-valued function.

An important special case is the gamma process with stationary increments, which is defined as a gamma process with shape function $at > 0$, $t \geq 0$, and scale parameter $u > 0$. The corresponding probability density function of $X(t)$ is then given by

$$f_{X(t)}(x) = \text{Ga}\left(x \,|\, at, u\right). \tag{5}$$

Due to the stationarity, both the mean value and the variance of the deterioration are linear in time; that is,

$$E(X(t)) = \frac{at}{u}, \quad \text{Var}(X(t)) = \frac{at}{u^2}.$$

A stochastic process has stationary increments if the probability distribution of the increments $X(t+h) - X(t)$ depends only on h for all $t, h \geq 0$.

A very useful property of the gamma process with stationary increments is that the gamma density in

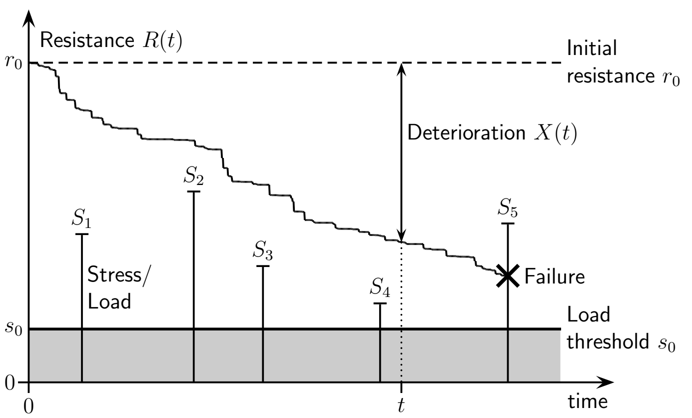

Figure 1: Stochastic processes of deterioration and stress/load.

Eq. (5) transforms into an exponential density if $t = a^{-1}$. When the unit-time length is chosen to be a^{-1}, the increments of deterioration are exponentially distributed with mean u^{-1} and the probability of failure in unit time i reduces to a shifted Poisson distribution with mean $1 + u(r_0 - s)$ (van Noortwijk et al., 1995) for $i = 1, 2, 3, \ldots$ Note that a shifted Poisson distribution is defined for $i = 1, 2, \ldots$ rather than for $i = 0, 1, \ldots$ A physical explanation for the appearance of the Poisson distribution is that it represents the probability that exactly i exponentially distributed jumps with mean u^{-1} cause the component to fail; that is, cause the cumulative amount of deterioration to exceed $r_0 - s$.

The unit time for which the increments of deterioration are exponentially distributed, facilitates the algebraic manipulations considerably and, moreover, often results in a very good approximation of the optimal design or maintenance decision, especially when the expected cost must be calculated.

Although it is not the purpose of this paper to give a rigourous mathematical treatise on the gamma process, we would like to explain the reader one important property of the gamma process, namely that it is a jump process. The key for showing this lies in the technique of Laplace transforms. In doing so, we first give the Laplace transform of a compound Poisson process representing jumps with intensity μ having random size. Then, we shall show that the Laplace transform of the gamma process can be rewritten in the same form as the Laplace transform of the compound Poisson process. For convenience, the proof is given for a compound Poisson process and a gamma process with stationary increments.

Let the continuous-time stochastic deterioration process $\{X(t), t \geq 0\}$ be a compound Poisson process given by $X(t) = \sum_{i=1}^{N(t)} D_i$, where

1. the number of jumps $\{N(t),\, t \geq 0\}$ is a Poisson process with jump intensity μ,
2. the jumps $\{D_i,\, i = 1, 2, \ldots\}$ are independent and identically distributed random quantities having distribution $\Pr\{D \leq \delta\} = F_D(\delta)$,
3. the process $\{N(t),\, t \geq 0\}$ and the sequence $\{D_i,\, i = 1, 2, \ldots\}$ are independent.

The Laplace transform of the compound Poisson process $X(t)$ is

$$E\left(e^{-sX(t)}\right) = \exp\left\{\mu t \left(E(e^{-sD}) - 1\right)\right\}. \tag{6}$$

According to De Finetti (1975, Chapter 8), this Laplace transform can be rewritten as

$$E\left(e^{-sX(t)}\right) = \exp\left\{\mu t \int_{\delta=0}^{\infty} \left(e^{-s\delta} - 1\right) dF_D(\delta)\right\} =$$

$$= \exp\left\{t \int_{\delta=0}^{\infty} \left(e^{-s\delta} - 1\right) \left[-dQ(\delta)\right]\right\}, \tag{7}$$

where $s > 0$ and

$$Q(\delta) = \mu\,[1 - F_D(\delta)] = \mu \int_{z=\delta}^{\infty} f_D(z)\,dz$$

represents the intensity of jumps whose magnitude is larger than δ for $\delta > 0$. The negative derivative of $Q(\delta)$, $q(\delta) = -Q'(\delta)$, is also called the Lévy measure of $\{X(t), t \geq 0\}$. The measure $q(\delta) = \mu f_D(\delta)$ is a positive measure, but not a probability measure because $\int_{\delta=0}^{\infty} q(\delta)\,d\delta = \mu \neq 1$. Note that the expected number of jumps of all sizes per unit time (jump intensity) is finite; that is, $Q(0) = \mu$ is finite. Indeed, for a compound Poisson process, the number of jumps in any time interval is finite with probability one. Furthermore, the expected value of $X(t)$ is also finite and has the value

$$E(X(t)) = t \int_{\delta=0}^{\infty} \delta\,[-dQ(\delta)] = \mu E(D)t.$$

On the other hand, the Laplace transform of $X(t)$ being a stationary gamma process is

$$\begin{aligned} E\left(e^{-sX(t)}\right) &= \left[\frac{u}{u+s}\right]^{at} = \\ &= \exp\left\{at \int_{\delta=0}^{\infty} \left(e^{-s\delta} - 1\right) \frac{e^{-u\delta}}{\delta}\,d\delta\right\} = \\ &= \exp\left\{t \int_{\delta=0}^{\infty} \left(e^{-s\delta} - 1\right) [-dQ(\delta)]\right\}, \end{aligned} \quad (8)$$

where $s > 0$ and

$$Q(\delta) = a \int_{z=\delta}^{\infty} \frac{e^{-uz}}{z}\,dz = aE_1(u\delta)$$

represents the intensity of jumps whose magnitude is larger than δ for $\delta > 0$ (see Dufresne et al., 1991) and the exponential integral is given by

$$E_1(x) = \int_{t=x}^{\infty} \frac{e^{-t}}{t}\,dt.$$

The Lévy measure of the gamma process is $q(\delta) = -Q'(\delta) = a\delta^{-1}e^{-u\delta}$. Because $\int_{\delta=0}^{\infty} q(\delta)\,d\delta = \infty$, this measure is infinite. Hence, the expected number of jumps of all sizes per unit time (jump intensity) is infinite as well; that is,

$$Q(0) = \lim_{\delta \downarrow 0} Q(\delta) = \infty.$$

Indeed, for a gamma process, the number of jumps in any time interval is infinite with probability one. Nevertheless, $E(X(t))$ is finite, as the majority of jumps are extremely small:

$$E(X(t)) = t \int_{\delta=0}^{\infty} \delta dQ(\delta) = at \int_{\delta=0}^{\infty} \delta \frac{e^{-u\delta}}{\delta} d\delta = \frac{at}{u}.$$

The agreement between Eqs. (7) and (8) shows us that the gamma process indeed is a jump process. As a matter of fact, the gamma process can be regarded as a limit of a compound Poisson process in the following manner. Let the probability density function of the jump sizes be a gamma distribution with shape parameter $v > 0$ and scale parameter $u > 0$ and let the Poisson jump intensity be $\mu = a\Gamma(v)/u^v$. According to Dufresne et al. (1991), the Laplace transform of this compound Poisson process is

$$\begin{aligned} E\left(e^{-sX(t)}\right) &= \\ &= \exp\left\{at\frac{\Gamma(v)}{u^v} \int_{\delta=0}^{\infty} \left(e^{-s\delta} - 1\right) \frac{u^v\delta^{v-1}}{\Gamma(v)} e^{-u\delta}\,d\delta\right\}. \end{aligned} \quad (9)$$

Clearly, the Laplace transform of the compound Poisson process with jump intensity $a\Gamma(v)/u^v$ and jump sizes being gamma distributed with shape parameter v and scale parameter u approaches the Laplace transform of a gamma process with shape function at and scale parameter u as v tends to zero from above.

Because deterioration should preferably be monotone, we can therefore best choose the deterioration process to be a compound Poisson process or a gamma process. In the presence of inspection data, the advantage of the gamma process over the compound Poisson process is evident: measurements generally consist of deterioration increments rather than of jump intensities and jump sizes.

3 STOCHASTIC LOAD PROCESS

In this section, the design stress is replaced by a stochastic process of random loads occurring at random times in the same way as it was done by Ellingwood and Mori (1993). We adopt their assumption of the loads occurring according to a Poisson process. The only difference is that we consider load threshold exceedances rather than the loads themselves, because the Poisson assumption is then better justified.

For probabilistic design, the distribution of extreme loads (such as water levels, discharges, waves, winds, temperatures and rainfall) is of considerable interest. For example, the probability distribution of high sea water levels is important for designing sea dikes. In order to determine the probability distribution of extreme load quantities, we can use extreme-value theory. Extreme-value analysis can be based on either maxima in a given time period, or on peaks over a given threshold. The advantage of using peaks over threshold rather than maxima is the availability of more observations. Using only maxima generally leads to loss of information.

In the peaks-over-threshold method, all load values larger than a certain threshold are considered. The differences between these values and a given threshold are called exceedances over the threshold. These exceedances are typically assumed to have a generalised Pareto distribution.

A random variable Y has a generalised Pareto distribution with scale parameter $\sigma > 0$ and shape parameter c if the probability density function of Y is given by

$$\mathrm{Pa}(y|\sigma, c) = \begin{cases} \dfrac{1}{\sigma}\left[1 - \dfrac{cy}{\sigma}\right]_+^{\frac{1}{c}-1}, & c \neq 0, \\ \dfrac{1}{\sigma}\exp\left\{-\dfrac{y}{\sigma}\right\}, & c = 0, \end{cases} \tag{10}$$

where $[y]_+ = \max\{0, y\}$ for $y > 0$. The survival function is defined by

$$\Pr\{Y > y\} = \bar{F}_Y(y|\sigma, c) = \begin{cases} \left[1 - \dfrac{cy}{\sigma}\right]_+^{\frac{1}{c}}, & c \neq 0, \\ \exp\left\{-\dfrac{y}{\sigma}\right\}, & c = 0. \end{cases}$$

The range of y is $0 < y < \infty$ for $c \leq 0$ and $0 < y < \sigma/c$ for $c > 0$. The case $c = 0$, which is the exponential distribution, is the limiting distribution as $c \to 0$. The generalised Pareto distribution mathematically arises as a class of limit distributions for exceeding a certain threshold, as the threshold increases toward the distribution's right tail. The conditions that must be satisfied in order to assure the existence of such limit distributions are rather mild. For the mathematical details, we refer to Pickands III (1975) and De Haan (1990).

Apart from the probability distribution of load threshold exceedances, the stochastic process of the occurrence times of these load exceedances needs to be specified. In doing so, we define the load threshold exceedance to be $Y = S - s_0 \geq 0$ with threshold s_0. Threshold exceedances $Y_1, \ldots, Y_n$ are assumed to be mutually independent and to have a generalised Pareto distribution, where $Y_i = S_i - s_0$, $i = 1, \ldots, n$. In most hydrological applications, the occurrence process of exceedances of large thresholds can be regarded as a Poisson process (see, e.g., Buishand, 1989). To overcome the problem of dependence between successive threshold exceedances in very small time periods, threshold exceedances are defined to be the peaks in *clusters* of threshold exceedances. Approximating the occurrence process of threshold exceedances by a Poisson process is supported by asymptotic extreme-value theory (Leadbetter, 1983).

The Poisson assumption implies that the probability that exactly n load threshold exceedances occur in time interval $(0, t]$ can be written as

$$\Pr\{N(t) = n\} = \frac{(\lambda t)^n}{n!}\exp\{-\lambda t\}, \tag{11}$$

$n = 0, 1, 2, \ldots$ for all $t \geq 0$; that is, the random quantity $N(t)$ follows a Poisson distribution with parameter λt. This parameter equals the expected number of threshold exceedances in time period $(0, t]$. The Poisson process $\{N(t), t \geq 0\}$ with $N(0) = 0$ has independent increments. Because the thresholds are exceeded according to a Poisson process, the inter-occurrence time of threshold exceedances has an exponential distribution with mean λ^{-1}:

$$\Pr\{T \leq t\} = F_T(t|\lambda) = 1 - \exp\{-\lambda t\}. \tag{12}$$

4 COMBINED DETERIORATION AND LOAD

Now we have defined both the stochastic process of deterioration and the stochastic process of load, the question arises how we can best combine them. In this respect, we also follow the approach of Ellingwood and Mori (1993). On the basis of a useful property of the Poisson process, they proposed an elegant way to combine stochastic load with a resistance decreasing over time. Mori and Ellingwood (1993) claim that the deterioration can be treated as deterministic, because "its variability was found to have a second-order effect on the structural reliability". This conclusion is based on Mori and Ellingwood (1994) in which they modelled the uncertainty in the deterioration process by posing a probability distribution on the damage growth rate a/u of the power law in Eq. (4). However, in Section 5, we will show this conclusion is not always justified.

In order to combine the load events with the deteriorating resistance, Mori and Ellingwood (1993) assumed that the duration of extreme load events is generally very short and is negligible in comparison with the service life of a structure. Therefore, they regard the load threshold exceedances as a series of pulses having an "infinitely small" duration (see Figure 1). As a consequence, we may then suppose that the resistance does not degrade during the occurrence of an extreme load event. Furthermore, they assume that failure can only occur during exceedances of the load threshold.

Consider the times at which changes of the Poisson process occur in time interval $[0, t]$ (in terms of threshold exceedances of the load); i.e., consider

$$Y_i = \sum_{h=1}^{i} T_h, \quad i = 1, \ldots, n,$$

where the inter-occurrence times $T_1, \ldots, T_n$ are independent, identically distributed, random quantities having an exponential distribution with mean λ^{-1}, and note that $0 \leq Y_1 \leq Y_2 \leq \cdots \leq Y_n \leq t$. Karlin and Taylor (1975, Pages 126-127) show us that $Y_1, \ldots, Y_n$ can be regarded as a set of order statistics of size n associated with a sample of n independent uniformly distributed random quantities on the interval $[0,t]$. The conditional probability distribution of the occurrence time of a load threshold exceedance, given one exceedance of the load threshold occurred in $(0,t]$, directly follows from the memoryless property of the exponential distribution: it is a uniform distribution on the interval $[0,t]$.

The conditional probability of no failure in time interval $(0,t]$, when n independent threshold exceedances of the load Y are given with cumulative distribution function $\Pr\{Y \leq y\} = F_Y(y)$, can now be formulated as (Karlin and Taylor, 1975, Page 180)

$$\begin{aligned}&\Pr\{\text{no failure in } (0,t] | N(t) = n\} = \\ &= E\left(\left[\int_{u=0}^{t} \frac{F_Y\left([r_0 - s_0] - X(u)\right)}{t}\, du\right]^n\right), \quad (13)\end{aligned}$$

where $s_0 \leq r_0$ and the expectation is defined with respect to the stochastic process $\{X(t), t > 0\}$. This result was obtained by Ellingwood and Mori (1993) for deterministic deterioration and extended by Mori and Ellingwood (1994) for deterioration with a random damage growth rate. Mathematically, the last step follows by conditioning on the sample paths of the gamma process, using the fact that the load threshold exceedances are independent, and applying the law of total probabilities with respect to the sample paths of the gamma process. Because failure can only occur at load threshold exceedances, and these exceedances are independent and uniformly distributed on $[0,t]$, we primarily can focus solely on these n exceedances. Eq. (13) generalises the lifetime model of van Noortwijk and Klatter (1999) in which the load was assumed to be equal to the constant design stress s.

Invoking the law of total probabilities with respect to the number of load threshold exceedances, the probability of no failure in time interval $(0,t]$ or survival probability can finally be written as

$$\begin{aligned}&\bar{F}(t) = 1 - F(t) = \Pr\{\text{no failure in } (0,t]\} = \quad (14) \\ &= \sum_{n=0}^{\infty} \Pr\{\text{no failure in } (0,t] | N(t) = n\} \Pr\{N(t) = n\} \\ &= E\left(\exp\left\{-\lambda \int_{u=0}^{t} \bar{F}_Y\left([r_0 - s_0] - X(u)\right)\, du\right\}\right).\end{aligned}$$

The cumulative distribution function of the lifetime is:

$$F(t) = 1 - E\left(\exp\left\{-\int_{u=0}^{t} k(R(u))\, du\right\}\right), \quad (15)$$

where $R(t) = r_0 - X(t)$ and the physics-based expression

$$k(r) = \lambda \bar{F}_Y(r - s_0) = \lambda \Pr\{S > r\} \quad (16)$$

can be interpreted as a killing rate in the sense of Wenocur (1989). In our stress-strength model, the killing rate is exactly the frequency of the stress $S = s_0 + Y$ exceeding the strength r. Hence, Eq. (16) gives a nice justification of Wenocur's definition of the killing rate. By differentiating Eq. (15), the lifetime probability density function becomes

$$f_T(t) = E\left(k(R(t)) \exp\left\{-\int_{\tau=0}^{t} k(R(\tau))\, d\tau\right\}\right). \quad (17)$$

Wenocur (1989) considers a very interesting extension of a stationary gamma process for the deterioration state under two different failure modes. A system is said to fail either when its condition reaches a failure level or when a traumatic event (such as an extreme load) destroys it. Suppose that the traumatic events occur as a Poisson process with a rate (called the 'killing rate') which depends on the system's condition. The latter is a meaningful assumption, since the worse the condition, the more vulnerable it is to failure due to trauma.

The following two-dimensional integral remains to be solved numerically:

$$\begin{aligned}&E\left(\exp\left\{-\int_{\tau=0}^{t} k(R(\tau))\, d\tau\right\}\right) = \quad (18) \\ &\lim_{n \to \infty} \int_0^{\infty} \cdots \int_0^{\infty} \exp\left\{-\sum_{i=1}^{n} k\left(r_0 - x_i\right)\left(t_i - t_{i-1}\right)\right\} \\ &\quad \times f_{X(t_1),\ldots,X(t_n)}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n,\end{aligned}$$

where $t_i = (i/n)t$, $i = 0, \ldots, n$. This integral is a special case of the so-called Kac functional equation (Beghin et al., 2000; Wenocur, 1989) and can be solved numerically in two combined steps. The first step is to approximate the integral over time by applying numerical integration with respect to the time grid $0, t_1, t_2, \ldots, t_{n-1}, t_n$. The second step is to approximate the integral over the sample paths of the gamma process by applying Monte Carlo simulation with respect to the independent increments $X(t_1), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$.

Although we can approximate a gamma process with a limit of a compound Poisson process, it is not efficient to simulate a gamma process in such a way. This is because there are infinitely many jumps in each finite time interval. A better approach for simulating a gamma process is simulating independent increments with respect to very small units of time. Gamma-increment sampling is defined as drawing independent samples $\delta_i = x_i - x_{i-1}$ from the gamma density with shape function $v(t_i) - v(t_{i-1})$ and scale parameter u for every $i = 1, 2, \ldots, n$, where $t_0 = x_0 = 0$.

5 EXAMPLE: DEN HELDER SEA DEFENSE

For the purpose of illustration, we study the probability of failure of a dike section subject to crest-level decline. The dike section is part of the sea defense at Den Helder in the North-West of the Netherlands and was constructed as early as in 1775.

The failure mechanism that we regard is overtopping of the sea dike by waves. Both sea level and crest-level decline are considered random, whereas wave run-up is considered fixed. The wave run-up is represented by the height $z_{2\%}$ (this is the wave run-up level in metres, which is exceeded by 2% of the number of incoming waves). Furthermore, the effect of seiches, gust surges and gust oscillations is taken into account by the factor b_0 [m]. For the Den Helder sea defense, the wave run-up height is computed as $z_{2\%} = 7.52$ m and the effect of seiches and gusts as $b_0 = 0.2$ m. The Den Helder sea defense has a crest level of $h = 12.33$ m +NAP (normal Amsterdam level). Including the effects of waves, seiches and gusts, the initial resistance of the dike can be defined as $r_0 = h - z_{2\%} - b_0 = 4.61$ m +NAP.

To account for the temporal variability of the sea level, we assume that the occurrence of extreme sea levels can be modelled with a peaks-over-threshold distribution. In particular, we use the generalised Pareto distribution estimated by Philippart et al. (1995) for observed sea levels at Den Helder. They estimated the following parameter values: $\lambda = 0.5$, $s_0 = 2.19$, $\sigma = 0.3245$, and $c = 0.05465$. Using this generalised Pareto distribution, the design water level with an exceedance probability of 10^{-4} is 4.40 m +NAP.

Crest-level decline consists of a combination of settlement, subsoil consolidation, and relative sea-level rise. In van Dantzig (1956), the expected crest-level

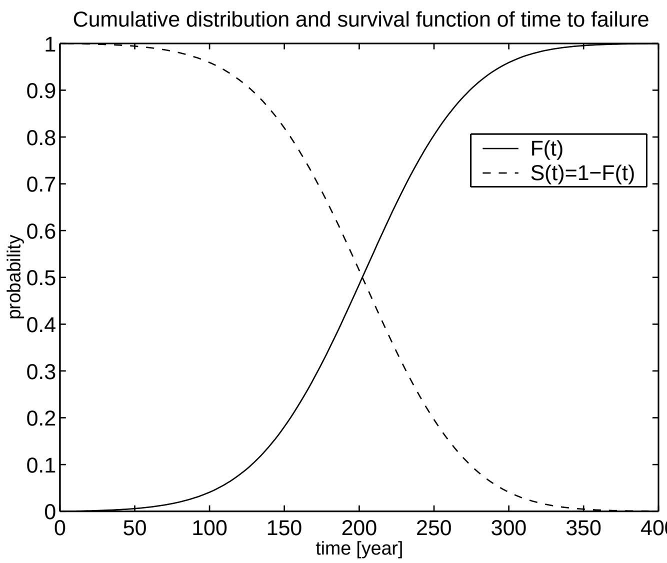

Figure 2: Cumulative distribution and survival function of the lifetime with $\mathrm{CV}(X(100)) = 0.3$.

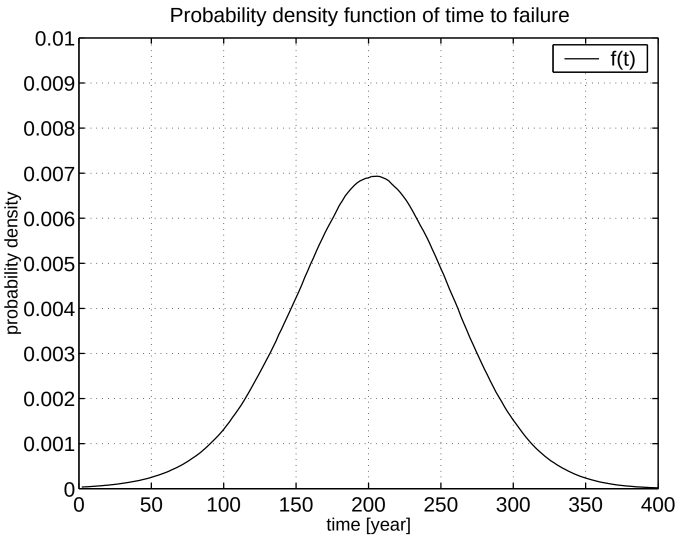

Figure 3: Probability density function of the lifetime with $\mathrm{CV}(X(100)) = 0.3$.

decline was assumed to be 0.7 m per century. We regard the stochastic process of crest-level decline as a stationary gamma process with mean $E(X(100)) = 0.7$ and a corresponding coefficient of variation of 0.3; that is,

$$E(X(100)) = \frac{a \cdot 100}{u} = 0.7,$$

$$\mathrm{CV}(X(100)) = \frac{\sqrt{\mathrm{Var}(X(100))}}{E(X(100))} = \frac{1}{\sqrt{a \cdot 100}} = 0.3.$$

Solving these equations for a and u leads to the parameter values $a = 0.1111$ and $u = 15.8730$. The 5th and 95th percentile of $X(100)$ are 0.39 m and 1.08 m, respectively.

In Figure 2, the cumulative distribution function and the survival function are shown based on Eq. (15). In this computation, the number of gamma-process sample paths is 10,000 with numerical integration

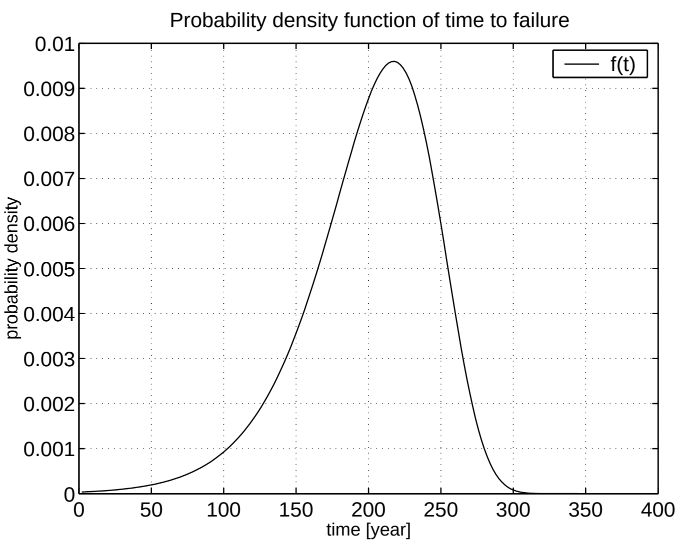

Figure 4: Probability density function of the lifetime with deterministic deterioration.

step size of $t_i - t_{i-1} = 2$ years for all i. The probability density function of the lifetime in Eq. (17) is graphed in Figure 3. In order to investigate the sensitivity of the probability density function of the time to failure with respect to the uncertainty in the deterioration, we also computed Eq. (17) with a deterministic deterioration. Obviously, Figure 4 shows that the uncertainty in the deterioration really matters in quantifying the uncertainty in the lifetime.

6 CONCLUSIONS

In this paper, a method is presented to compute the time-dependent reliability of structures where both the deteriorating resistance and the fluctuating load are modelled as stochastic processes. The stochastic process of deterioration is modelled as a gamma process having independent gamma-distributed increments with identical scale parameter. The stochastic process of load is modelled as a combination of a peaks-over-threshold distribution (such as the generalised Pareto distribution) and a Poisson process for the threshold exceedances. It is shown that the cumulative distribution function of the time to failure (time at which the resistance drops below the load) can be formulated as a Kac functional equation. This equation can be solved by applying a combination of numerical integration and simulating sample paths of the gamma process. The advantages of the proposed method are that a gamma process properly models the monotonic behaviour of ageing and that peaks-over-threshold distributions fit in well with extreme-value statistics.

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