A study towards the application of Markovian deterioration processes for bridge maintenance modelling in the Netherlands

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ABSTRACT: Infrastructures in the Netherlands are rated according to a discrete condition scheme and the results are registered in an extensive database. The paper discusses the application of a continuous-time Markov process for modelling deterioration and introduces new results for optimizing periodic inspections and preventive maintenance. The focus of this discussion is on the condition of superstructures of concrete highway bridges in the Netherlands for which 19 years of inspection data are available.

1 INTRODUCTION

Inspection and maintenance planning of bridges in the Dutch national road system is one of the responsibilities of the Civil Engineering Division of the Ministry of Transport, Public Works and Water Management. In recent years, several studies have been done to assess the rate of deterioration in concrete bridges. It has proven to be difficult to reach a consensus on parameters of physical models, e.g. chloride ingress, and on how these physical processes affect the load bearing capacity of a bridge. A statistical approach was presented in van Noortwijk and Klatter (2004), where lifetime distributions were used to model the uncertain lifetime of bridge elements. Weibull distributions were fitted to data of current and replaced bridge ages.

As is done in many other countries, infrastructures in the Netherlands are inspected on a regular basis and the observed condition of the structure is given on a discrete scale with a finite number of condition states. An important property of these inspections is that they are performed periodically as opposed to continuously. Because of this, the exact times of transitions between states are not observed. Also, the qualitative nature of condition states makes is practically impossible to precisely identify the exact moment of a transition. The distinction between states, in most condition rating systems, is not strict and the interpretation of each condition state is subjective.

Current bridge management systems commonly use discrete-time Markov processes (Markov chains) to model the uncertain deterioration in time. This probabilistic approach is well suited for modelling deterioration on a discrete condition scale. Also, most calculations that are involved are relatively straightforward and computationally efficient. Nonetheless, concerns about the suitability of their use have increased in recent years. Aging is a property which is notably missing in modelling deterioration using a Markov chain. This is a property in which the probability of the structure performing a state transition increases in time. It can be included by using a probability distribution with increasing failure rate for the uncertain time spent in each condition state. For example, Kleiner (2001) and Mishalani and Madanat (2002) have suggested the use of a Weibull distribution for this purpose. However, one needs to know the exact duration between transitions to be able to determine the parameters of the Weibull distribution. As explained earlier, this information is not available. However, this approach has the nice property that transitions are not restricted to discrete moments in time, but occur on a continuous-time scale.

In the present paper, a continuous-time Markov process with exponential waiting times is applied to the bridge inspection data in the Netherlands. Section 2 discusses the inspection data, section 3 presents the notation used throughout the remainder of the paper, and section 4 introduces the continuous-time Markov process for modelling deterioration. The latter section discusses the maximum likelihood estimator for the mean rate of deterioration and presents new analytical results for an inspection model, which allows for preventive maintenance. Two example applications are presented, in which the optimal inspection interval is calculated based on the expected average costs

Table 1. Unofficial condition rating scheme for infrastructures in the Netherlands.

State	Damage rating	Structure rating
0	none	perfect
1	initiation	very good
2	minor	good
3	multiple/serious	reasonable
4	advanced/grave	mediocre
5	threat to safety/functionality	bad
6	extreme danger	very bad

per year. Finally, since the proposed continuous-time model assumes linear deterioration in time, a discrete-time semi-Markov process is presented in Section 5 and a comparison of results is made.

2 BRIDGE INSPECTION DATA

Since late 1985, the results of inspections of infrastructures in the Netherlands are registered in an electronic database. This database is primarily used as a central collection point of all information regarding individual structures. This information includes, amongst others, the year of construction, geographical location, technical drawings, record of damages, and inspection history.

Each structure is divided into parts, each part into main components, and each main component into elements. For example: each part in a bridge contains a superstructure, which is one of a number of main components. The beams and the road surface are individual elements of the bridge superstructure. During an inspection, each damage to the bridge is registered and given a condition rating from 0 to 6. Based on these damages and their severity, the main components and the structure as a whole are also given a rating. Table 1 shows the unofficial interpretation of this rating scheme, which is used by inspectors. It is an unofficial scheme as there are no strict regulations on condition assessment of infrastructures in the Netherlands.

The database includes a total of 5986 registered inspection events for 2473 individual superstructures. Ignoring the time between the construction of the bridge and the first inspection, there are over 3500 registered transitions between condition states. Transitions between the year of construction and the first registered inspection are ignored, because most bridges were constructed well before 1985 and it would be incorrect to assume that they have not had some kind of repair during this time. Starting from the first registered inspection, the times of successive inspections and their rating results are known. Table 2 shows the count, up to the summer of 2004, of transitions between each possible pair of condition states. The results of new inspections are added to the

Table 2. Count of transitions between condition states for superstructures of concrete bridges.

		To						
		0	1	2	3	4	5	6
From	0	520	134	327	111	36	7	0
	1	270	128	222	97	36	7	0
	2	284	101	368	193	61	9	5
	3	94	33	119	131	42	3	1
	4	16	14	42	50	17	7	0
	5	7	3	4	4	3	0	1
	6	1	1	0	3	1	0	0

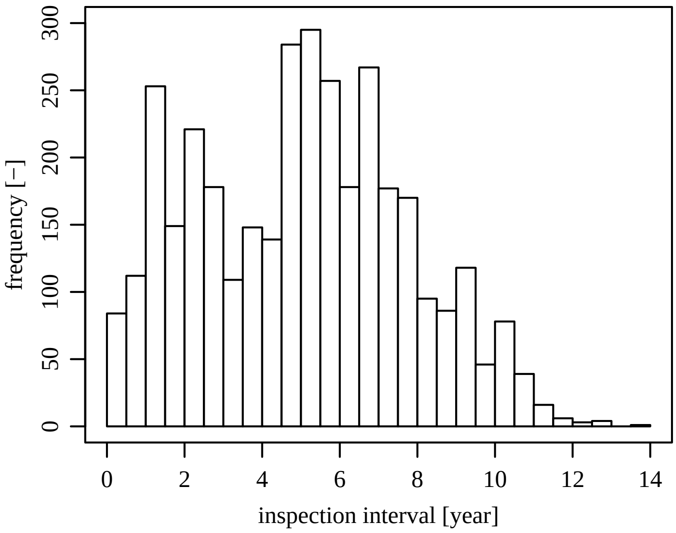

Figure 1. Histogram of inspection interval times.

database almost on a daily basis, therefore the number of registered transitions increases continuously. From Table 2 it is observed that states 5 and 6 rarely occur in the database. In this paper, these states are combined into state 5 to form a single state representing conditions 5 and worse. Also, the transitions representing a quality improvement will be ignored in the remainder of this paper. Due to a lack of detailed information, it is not possible to clearly identify and remove only transitions due to maintenance.

Figure 1 shows a histogram of the times between registered inspection events. Each bin represents a time period of 6 months and the height of the bin shows the number of inspection interval times which fall into each period. The longest time between two inspections is 167 months and the average time is about 60 months or 5 years. The histogram suggests that the distribution of the time between inspections is bimodal with peaks around 2 and 6 years.

3 NOTATION

The reader is assumed to be familiar with the theory of Markov processes. The notation used in the present paper is similar to that used in e.g. Ross (1970). An object is assumed to be in any condition state given by the set $\mathcal{S} = \{0, 1, \ldots, m\}$. Let $\{X(t),\ t \geq 0\}$ denote the state of the Markov process at time t and J_n the state of the object after $n = 0, 1, 2, \ldots$ transitions. Note that if $N(t)$ is the number of transitions up to time t, then $X(t) = J_{N(t)}$. The transition probabilities for the Markov chain are denoted by $P_{ij} = \Pr\{J_{n+1} = j \,|\, J_n = i\}$ and the transition probability matrix by $\boldsymbol{P} = ||P_{ij}||$. For a semi-Markov process, let $F_{ij}(t) = \Pr\{T \leq t \,|\, J_n = i, J_{n+1} = j\}$ denote the cumulative probability distribution of the transition time given that the object performs a transition from state i to j. The probability of moving from state i to j in an amount of time less than or equal to t is given by the product $Q_{ij}(t) = P_{ij}F_{ij}(t)$.

4 CONTINUOUS-TIME MARKOV PROCESS

It is assumed that the waiting time in each condition state is exponential with rate $\lambda > 0$, i.e. $T \sim \text{Exp}(\lambda)$ and $F_T(t) = 1 - \exp(-\lambda t)$, and that the structure moves into the next worse condition at each transition with probability one. With these assumptions, the transitions occur according to a Poisson process. The probability of performing n transitions during a time period of length t is therefore given by the Poisson distribution:

$$\Pr\{N(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad \text{for } n = 0, 1, 2, \ldots \qquad (1)$$

For superstructures, many observations of the number of transitions during a period of time are available from the inspection database. If there are k observations $(t_1, n_1), (t_2, n_2), \ldots, (t_k, n_k)$, then the likelihood of the data is given by

$$\Pr\{N(t_1) = n_1, \ldots, N(t_k) = n_k\} = \prod_{i=1}^{k} \frac{(\lambda t_i)^{n_i}}{n_i!} e^{-\lambda t_i}. \qquad (2)$$

The maximum likelihood estimator for λ is

$$\hat{\lambda} = \sum_{i=1}^{k} n_i \Big/ \sum_{i=1}^{k} t_i. \qquad (3)$$

Using the data from the inspections, an estimate for the rate is obtained: $\lambda = 0.1793$. The mean waiting time in each state is therefore approximately $1/\lambda = 5.577$ years.

Let $S_n = \sum_{i=1}^{n} T_i$, where T_i is the waiting time between transitions, be the time it takes for the process to perform n transitions. The sum of n exponential random variables with the same scale parameter $\lambda > 0$ has a gamma distribution. The probability density function is given by

$$f_{S_n}(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t} \quad \text{for } t \geq 0, \qquad (4)$$

where $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$ is the gamma function. For $n = 1, 2, \ldots$, this distribution is also called the Erlang distribution and $\Gamma(n) = (n-1)!$. The result for the superstructure inspection data is presented in Figure 6.

4.1 Inspection model

This relatively simple deterioration process can now be used to optimize inspection and maintenance decisions. The classic approach, see e.g. Ross (1970), would be to use a semi-Markov decision process to determine the policy with the lowest expected (discounted) costs. The solution is obtained by calculating successive policies until the expected costs can be no further minimized. This is the so-called ‘policy improvement algorithm’. In this paper, a less common, but far easier approach is introduced.

Assume the following: the bridges are inspected periodically at regular time intervals and a repair is performed based on the condition of the structure at the time of inspection. Let the threshold for preventive repair (before failure) be state r ($0 < r < m$) and for corrective repair (after failure) state s ($r < s \leq m$). Repairs are assumed to be instantaneous and performed immediately after inspection. Also, repairs are assumed to be perfect, therefore the structure is as good as new after a repair. Using renewal theory, the expected costs per unit time are given by the ratio of the expected costs per cycle over the expected cycle length. A cycle starts after the construction of the object or after a perfect repair and ends when either a preventive or corrective repair is performed.

To determine the expected time at which a cycle is ended by a repair, the probability of both a preventive and a corrective repair after an inspection must be calculated. The following equivalence holds:

$$X(t) \leq n \Leftrightarrow S_n \geq t. \qquad (5)$$

In words: if the state of the object is less than or equal to n at time t, the time required to perform n transitions is greater than or equal to t. Let $\tau > 0$ be the inspection interval and $k = 0, 1, 2, \ldots$ such that $k\tau$ represents the time of the k-th inspection. A preventive repair is performed at time $k\tau$ if the state of the object during the previous inspection at time $(k-1)\tau$ had not yet reached state r, but is greater than or equal to r and less than the failure state s at time $k\tau$. The probability of this event is

$$\Pr\{\text{preventive repair in period } ((k-1)\tau, k\tau]\} = \Pr\{X((k-1)\tau) < r, r \leq X(k\tau) < s\}.$$

Using the relationship in Equation (5), this probability becomes

$$\Pr\{(k-1)\tau < S_r \le k\tau, S_s > k\tau\} = \sum_{j=0}^{s-r-1} \frac{(\lambda k\tau)^{j+r}}{(j+r)!} e^{-\lambda k\tau} \left[1 - I_{1-\frac{1}{k}}(r, j+1)\right], \quad (6)$$

where $I_x(a,b) = \int_{\phi=0}^{x} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \phi^{a-1}(1-\phi)^{b-1} d\phi$ is the incomplete beta function for $0 \le x \le 1, a > 0$ and $b > 0$, see e.g. Abramowitz and Stegun (1965). The derivation of this result is given in the appendix at the end of this paper. For a corrective repair, the state at the previous inspection was again less than r, but is greater than or equal to the failure state s at time $k\tau$:

$$\Pr\{\text{corrective repair in period } ((k-1)\tau, k\tau]\} = \Pr\{X((k-1)\tau) < r, X(k\tau) \ge s\}.$$

Again, using Equation (5), this becomes

$$\begin{aligned}
&\Pr\{S_r > (k-1)\tau, (k-1)\tau < S_s \le k\tau\} \\
&= \Pr\{(k-1)\tau < S_r < S_s \le k\tau\} \\
&= \Pr\{(k-1)\tau < S_r \le k\tau\} \\
&\quad - \Pr\{(k-1)\tau < S_r \le k\tau, S_s > k\tau\} \\
&= \Pr\{S_r > (k-1)\tau\} - \Pr\{S_r > k\tau\} \\
&\quad - \Pr\{(k-1)\tau < S_r \le k\tau, S_s > k\tau\} \\
&= P(\lambda k\tau, r) - P(\lambda(k-1)\tau, r) \\
&\quad - \sum_{j=0}^{s-r-1} \frac{(\lambda k\tau)^{j+r}}{(j+r)!} e^{-\lambda k\tau} \left[1 - I_{1-\frac{1}{k}}(r, j+1)\right], \quad (7)
\end{aligned}$$

where $P(x,a) = \frac{1}{\Gamma(a)} \int_{t=0}^{x} t^{a-1} e^{-t} dt$ is the incomplete gamma function, see Abramowitz and Stegun (1965).

4.2 Cost optimization

Two cost models are presented here. One where failure of a structure is noticed immediately without the necessity of performing an inspection and one model where failure is not noticed until the next planned inspection. The latter model suits the case of superstructures very well, because physical failure never occurs and an inspection is required to assess the state of the superstructure.

The costs per cycle are the sum of the costs of all inspections and either a single preventive or a single corrective replacement. For the first model, the expected cycle costs are

$$\mathbb{E}[C] = \sum_{k=1}^{\infty} \Big[(kc_I + c_R)\Pr\{\text{PR in } ((k-1)\tau, k\tau]\} + ((k-1)c_I + c_F)\Pr\{\text{CR in } ((k-1)\tau, k\tau]\}\Big], \quad (8)$$

where PR is preventive repair, CR is corrective repair, and c_I, c_P and c_F are the costs of an inspection, preventive repair and a corrective repair respectively. The expected cycle length is

$$\mathbb{E}[I] = \sum_{k=1}^{\infty} \Big[k\tau \Pr\{\text{PR in } (k-1, k]\} + \sum_{n=(k-1)\tau+1}^{k\tau} n\Pr\{\text{CR in } (n-1, n]\}\Big]. \quad (9)$$

The summation over n from $(k-1)\tau + 1$ to $k\tau$ reflects the immediate identification of a failure.

For the second model, in which failure is not noticed until the next inspection, Equations (8) and (9) become

$$\mathbb{E}[C] = \sum_{k=1}^{\infty} \Big[(kc_I + c_R)\Pr\{\text{PR in } ((k-1)\tau, k\tau]\} + ((k-1)c_I + c_F)\Pr\{\text{CR in } ((k-1)\tau, k\tau]\} + \sum_{n=(k-1)\tau+1}^{k\tau} c_U(k\tau - n)\Pr\{\text{failure in } (n-1, n]\}\Big], \quad (10)$$

and

$$\mathbb{E}[I] = \sum_{k=1}^{\infty} \Big[k\tau \Pr\{\text{PR in } ((k-1)\tau, k\tau]\} + k\tau \Pr\{\text{CR in } ((k-1)\tau, k\tau]\}\Big], \quad (11)$$

where c_U are the costs of unavailability per unit time. This cost is added in Equation (10) as a penalty for leaving a structure in a failed state. The costs increase proportionally to the time that the superstructure is left in the failed state. Without this cost, the cheapest solution would be to not inspect at all, because the costs per unit time will decrease as the cycle length increases.

Two hypothetical examples are considered with fictitious costs presented in Table 3. Example A considers the case in which a failure is immediately detected and repaired without the need for an inspection. Costs

Table 3. Costs for two hypothetical examples.

	Example A	Example B
Failure detection	immediate	by inspection
Inspection (c_I)	€ 1000	€ 1000
Prev. repair (c_P)	€ 10000	€ 10000
Corr. repair (c_F)	€ 40000	€ 10000
Unavailability (c_U)	N/A	€ 2000

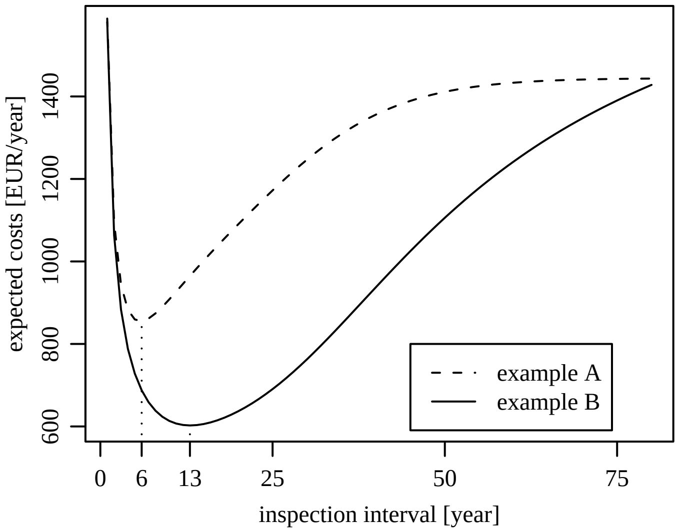

Figure 2. Expected average costs per year as a function of the inspection interval τ.

for corrective repair are four times the costs of a preventive repair, therefore it will be economically interesting to repair before failure occurs. Example B considers the other case, in which failure is not detected until the next inspection and a cost is incurred for each unit of time that the superstructure is in a failed state. Also, the cost for corrective repair is the same as for a preventive repair, therefore it will not be economically interesting to perform preventive maintenance. For both examples, the threshold for preventive repair is state $r = 3$ and the failed state is state $s = 5$. The unit time considered in both examples is one year.

The results for both examples are shown in Figure 2. Since corrective repair is expensive compared to preventive repair in example A, the inspection interval with lowest expected average costs per year is shorter than the same optimal value for example B: 6 years compared to 13 years for example B. With a mean time to preventive repair of about $r/\lambda = 3 \times 5.577 = 16.7$ years. This implies that about 3 inspections are performed for example A and a preventive repair is done after about 18 years. For example B, only about 2 inspections are performed and a repair is performed after about 26 years. As $\tau \to \infty$, the costs in example A converge to the costs of one inspection and a corrective repair, € 41000, divided by the expected lifetime of 28 years, which is approximately € 1460. The costs in example B do not converge, but increase every year due to the cost of unavailability.

The cost model presented in this section is easier to implement than the classic policy improvement algorithm. Instead of presenting the decision maker with a single optimal value, this approach results in a clear graphical presentation as is demonstrated by Figure 2. Also, the models can be adjusted for various situations. For example, Equations (8) to (11) can be adjusted to include discounting, see e.g. van Noortwijk et al. (1997), or to include the time of the first inspection as an extra decision variable. The latter extension has been demonstrated before by Jia and Christer (2002) and is useful when the thresholds for preventive and corrective maintenance are very close to each other. When the difference in costs for preventive or corrective replacement is large, this would result in a very short inspection interval, which would be unnecessary when the structure has just been taken into service. The model will determine the optimal combination of the time of first inspection and the subsequent periodic inspection interval.

5 MODEL COMPARISON

Because the choice for the continuous-time Markov process implicitly assumes a linear deterioration rate, a semi-Markov process is fitted to the data and the results are compared. In a semi-Markov process, the Markov property only holds at the transition times. The waiting time is modelled by a probability distribution other than the exponential distribution.

For semi-Markov processes, the probability of moving from state i to state j during a time period t is given by the transition probability function, see e.g. Ross (1970):

$$P_{jj}(t) = 1 - \sum_{k=0}^{m} \int_{x=0}^{t} [1 - P_{kj}(t-x)]\, dQ_{jk}(x),$$
$$P_{ij}(t) = \sum_{k=0}^{m} \int_{x=0}^{t} P_{kj}(t-x) dQ_{ik}(x), \quad i \neq j, \tag{12}$$

where $P_{ij}(t) = \Pr\{X(t) = j \,|X(0) = i\}$. The discrete-time semi-Markov process was defined by Howard (1971b) as an extension of Markov chains. By restricting the transition times to a discrete time scale, Equation (12) can be approximated for computational convenience by

$$P_{jj}(t) = 1 - \sum_{k=0}^{m} \sum_{x=1}^{t} [1 - P_{kj}(t-x)]\, q_{jk}(x),$$
$$P_{ij}(t) = \sum_{k=0}^{m} \sum_{x=1}^{t} P_{kj}(t-x) q_{ik}(x), \quad i \neq j, \tag{13}$$

where it is assumed that $t = 1, 2, \ldots$ and $q_{ij}(t) = \Pr\{T = t, J_n = i, J_{n+1} = j\}$. Obviously, $P_{jj}(0) = 1$ and $P_{ij}(0) = 0$ for $i \neq j$. The product $P_{kj}(t-x)q_{ik}(x)$ represents the probability of a transition from state i to k in a time x and subsequently moving from k to j in the remaining $t - x$ time.

For a sufficient number of observations, Billingsley (1961) has shown that the maximum likelihood estimator for the transition probability from state i to j

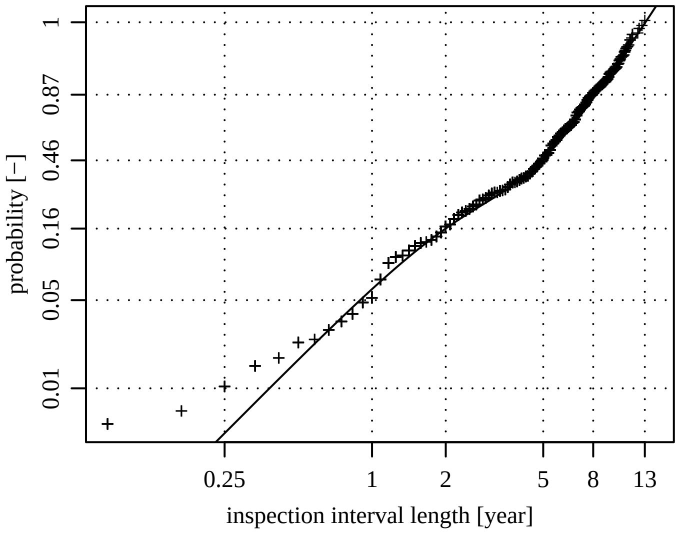

Figure 3. Weibull probability plot for inspection interval times (+) with a fitted 2-Weibull mixture distribution (solid line).

is simply given by $\hat{P}_{ij} = N_{ij} / \sum_{j=0}^{m} N_{ij}$, where N_{ij} is the number of observed transitions from i to j. Ignoring the transition counts in the lower triangular part of Table 2, the following transition matrix is obtained:

$$\boldsymbol{P} = \begin{bmatrix} 0.46 & 0.12 & 0.29 & 0.09 & 0.03 & 0.01 \\ 0 & 0.26 & 0.45 & 0.20 & 0.07 & 0.02 \\ 0 & 0 & 0.58 & 0.30 & 0.10 & 0.02 \\ 0 & 0 & 0 & 0.74 & 0.24 & 0.02 \\ 0 & 0 & 0 & 0 & 0.71 & 0.29 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (14)$$

To determine $q_{ij}(t)$, the probability distribution of the waiting time $F_{ij}(t)$ is needed. An informal analysis of the data suggests that the inspection interval length has a low dependence with the state the structure is in. For simplification, it is therefore assumed that the waiting time is independent of the condition state. The bimodal random waiting time T, as shown in Figure 1, is modelled by a 2-Weibull mixture distribution of which the probability density is given by

$$f_T(t) = p g_1(t) + (1-p) g_2(t), \quad (15)$$

where $0 \leq p \leq 1$ and

$$g_i(t) = (a_i/b_i)\,(t/b_i)^{a_i - 1} \exp\left\{-(t/b_i)^{a_i}\right\}$$

is the Weibull density function with shape parameter $a_i > 0$ and scale parameter $b_i > 0$. Figure 3 shows the maximum likelihood fit with parameters $p = 0.1999, a_1 = 1.9752, b_1 = 1.8199, a_2 = 3.0505$ and $b_2 = 6.7329$.

Using Equation (13), the relative frequency of states for superstructures of age t in the bridge network can be determined by

$$\Pr\{X(t) = j\} = \sum_{i=0}^{m} P_{ij}(t) \Pr\{X(0) = i\}.$$

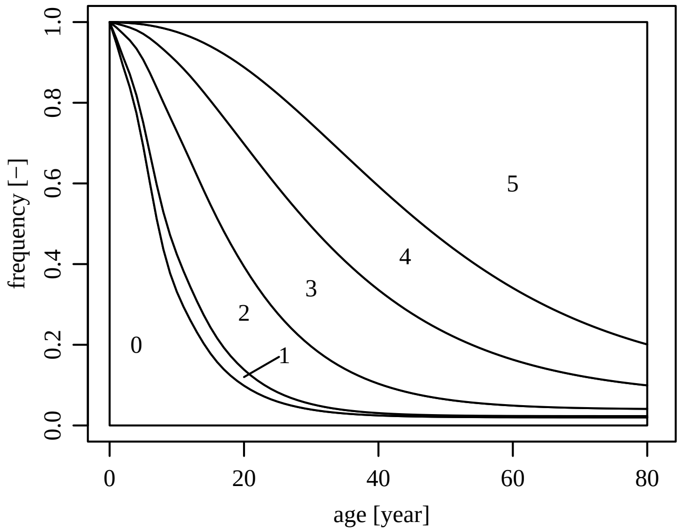

Figure 4. Relative frequency of condition states as a function of bridge age.

For $\Pr\{X(0) = 0\} = 1$, this result is shown in Figure 4 as a function of t, which is considered to be the age of the superstructure. This result shows that if all superstructures start in the initial state 0 and no maintenance would be performed, less than 3% of these would still be in the initial state and 40% would already be in state 5 by the age of 40 years. State 1 is very rare and the second column of the transition matrix (14) shows that objects in state 0 have only a small probability of moving into state 1 and objects in state 1 have a small probability of staying there. Deterioration is initially quite fast, but slows down after the structures go into state 3 and 4.

The expected state at time t can be easily obtained by calculating $\mathbb{E}[X(t)] = \sum_{j=0}^{m} j \cdot \Pr\{X(t) = j\}$. In Figure 5, this expectation is compared to the expected state of the continuous-time Markov process. Using Equations (1) and (4), the expected condition for the continuous-time Markov process is determined by:

$$\mathbb{E}[X(t)] = \sum_{j=0}^{m-1} j \cdot \Pr\{N(t) = j\} + m \cdot \Pr\{S_m \leq t\}.$$

The maximum likelihood estimate of the parameter λ is dominated by the observations of transitions between states 0, 1 and 2, which together constitute more than two thirds of all observations. Note that the expectation for the continuous-time Markov proces is initially linear, but eventually converges to the absorbing state 5. Although bridge deterioration is usually assumed to be non-linear, the result in Figure 5 is somewhat surprising. In this case, deterioration is faster in the early stages compared to later stages, whereas the most common assumption is the exact opposite.

Now, the first passage time for the discrete-time semi-Markov process is determined and compared to the equivalent result for the continuous-time Markov

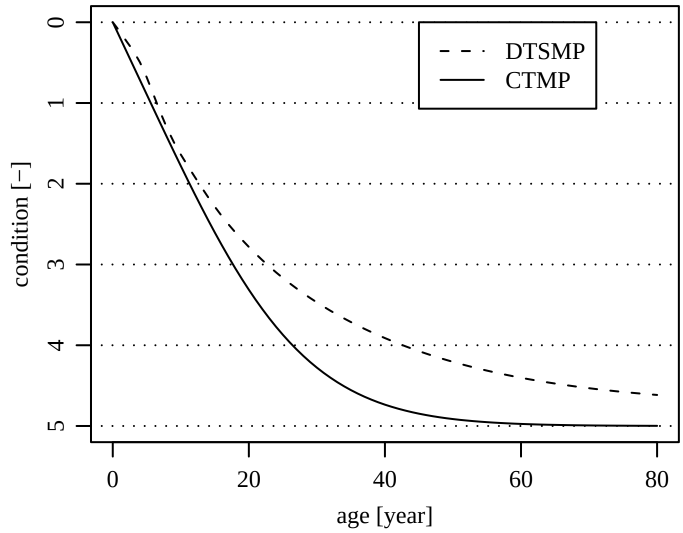

Figure 5. Expected state of superstructures determined by the discrete-time semi-Markov (DTSMP) and the continuous-time Markov processes (CTMP).

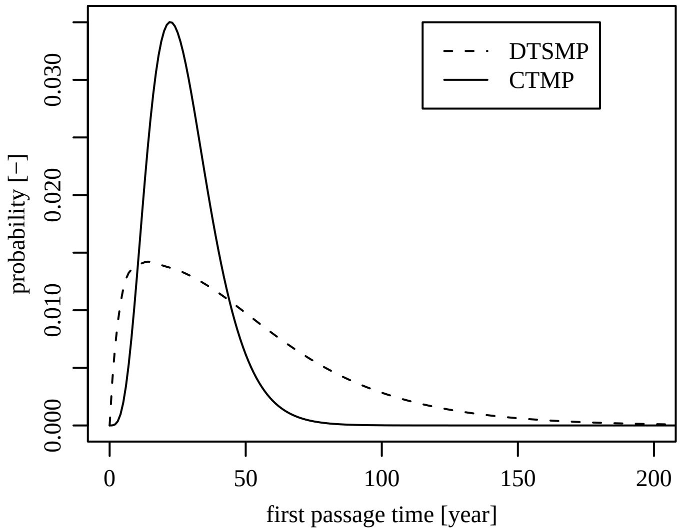

Figure 6. Probability density of first passage time from state 0 to 5 for the discrete-time semi-Markov process (DTSMP) and continuous-time Markov process (CTMP).

process. The first passage time is the first time the process reaches a selected state. The probability density of first reaching state j from state i after exactly n steps

$$f_{ij}(n) = \Pr\{J_n = j, J_{n-1} \neq j, \ldots, J_1 \neq j \,|\, J_0 = i\}$$

can be calculated using the following simple recursive equation; Howard (1971a):

$$f_{ij}(n) = \begin{cases} \sum_{k\neq j}^{m} P_{ik} f_{kj}(n-1), & n > 1, \\ P_{ij}, & n = 1. \end{cases} \quad (16)$$

The first passage time is the product of the number of transitions and the waiting time per transition if, as is assumed here, the waiting time T is independent of the transition. Let N be a discrete random variable representing the number of transitions to first passage and let $Z = NT$. The probability distribution of the product of these two random variables is given by

$$f_Z(z) = \sum_{n=1}^{\infty} n^{-1} f_N(n) f_T(z/n),$$

where $f_T(t)$ is the density of the waiting time given by Equation (15) and $f_N(n)$, for $n = 1, 2, \ldots$, is the probability function of the number of steps given by Equation (16). Figure 6 shows the result for the first passage time from state 0 to 5 and compares it to the same result for the continuous-time Markov process. The distribution for the discrete-time semi-Markov process has a very thick tail, which places the mean first passage time relatively high: approximately 47 years. The uncertainty in the first passage time for the continuous-time Markov process is much smaller and has almost no tail compared to the discrete-time Markov process. The expected lifetime in this case is approximately 28 years. Note that the expected time to failure is not the same as the time for the expected condition to reach state 5.

6 CONCLUSIONS

Traditionally, Markov chains have been used for modelling deterioration on a discrete condition scale. Semi-Markov processes have recently received interest in structural engineering, because they allow for the use of random waiting times in condition states. Preferably, the waiting times should be modelled by probability distributions with an increasing failure rate. However, experience with bridge inspection data in the Netherlands, shows that adequate information for proper fitting of the distribution parameters is unavailable. This is due to the periodicity of the inspections and the subjective rating systems generally used in visual condition evaluations.

A continuous-time Markov process provides a probabilistic framework for inspection and maintenance optimization, which is analytically tractable. The deterioration model can be easily fitted to the observations from the inspection database in the Netherlands. The cost model introduced in Section 4.2 provides an alternative approach to the classic policy iteration algorithm. Two examples have been presented: one where failure is observed immediately without an inspection and a second example where failure is not noticed until the next inspection.

An important restriction of the continuous-time Markov model for deterioration is the assumption that the mean waiting time is the same for all states. Further development could therefore include the possibility of defining λ_i for each state $i = 0, 1, \ldots, m-1$, such that the deterioration process can also be nonlinear. This will, amongst others, require different formulations for the likelihood of observations in Equation (2) and the probability of preventive and corrective repair in Equations (6) and (7) respectively.

From an organizational point of view, more research will also need to be done in order to better differentiate deterioration from maintenance in the data in Table 2. It is relatively sure that a transition from state 5 to 0 was due to maintenance. However, a transition from state 4 to 3 could also occur because of the inconsistency in condition rating, which is due to the subjective interpretation of the damage and the rating system by the inspector(s). Unfortunately, maintenance actions have not been registered in the same database as the inspections, but are kept decentralized. A strong effort will be required to filter those transitions from the data, which are really due to deterioration. Finally, the imperfection of the visual evaluation by inspectors could also be modelled.

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8 APPENDIX

The result in Equation (6) is obtained by splitting the probability and using the independence of the increments S_r and $S_s - S_r$:

$$\begin{aligned}
&\Pr\{(k-1)\tau < S_r \leq k\tau, k\tau - S_r < S_s - S_r\} \\
&= \Pr\{S_r \leq k\tau, k\tau - S_r < S_s - S_r\} - \\
&\qquad \Pr\{S_r \leq (k-1)\tau, k\tau - S_r < S_s - S_r\}.
\end{aligned}$$

Denoting the difference of these two probabilities by $A - B$, each of these can be calculated as follows:

$$\begin{aligned}
A &= \int_{\phi=0}^{k\tau} \int_{\theta=k\tau-\phi}^{\infty} f_{S_r}(\phi) f_{S_s-S_r}(\theta)\, d\theta\, d\phi \\
&= \int_{\phi=0}^{k\tau} f_{S_r}(\phi) \left[\int_{\theta=k\tau-\phi}^{\infty} f_{S_s-S_r}(\theta)\, d\theta \right] d\phi \\
&= \int_{\phi=0}^{k\tau} \frac{\lambda^r \phi^{r-1} e^{-\lambda\phi}}{(r-1)!} \left[\sum_{j=0}^{s-r-1} \frac{\lambda^j (k\tau-\phi)^j}{j!} e^{-\lambda(k\tau-\phi)} \right] d\phi \\
&= \sum_{j=0}^{s-r-1} \left[\frac{\lambda^{(j+r)}}{(j+r)!} e^{-\lambda k\tau} (k\tau)^{j+r-1} \cdot \right. \\
&\qquad \left. \cdot \int_{\phi=0}^{k\tau} \frac{(j+r)!}{(r-1)!j!} \left(1 - \frac{\phi}{k\tau}\right)^j \left(\frac{\phi}{k\tau}\right)^{r-1} d\phi \right].
\end{aligned}$$

With the substitution $\varphi = \frac{\phi}{k\tau}$, the beta function integrates out:

$$\begin{aligned}
A &= \sum_{j=0}^{s-r-1} \left[\frac{\lambda^{(j+r)}}{(j+r)!} e^{-\lambda k\tau} (k\tau)^{j+r-1} \cdot \right. \\
&\qquad \left. \cdot\, k\tau \int_{\varphi=0}^{1} \frac{(j+r)!}{(r-1)!j!} (1-\varphi)^j \varphi^{r-1} d\varphi \right] \\
&= \sum_{j=0}^{s-r-1} \frac{(\lambda k\tau)^{j+r}}{(j+r)!} e^{-\lambda k\tau}.
\end{aligned}$$

The same calculations can be used to derive the second part:

$$B = \sum_{j=0}^{s-r-1} \frac{(\lambda k\tau)^{j+r}}{(j+r)!} e^{-\lambda k\tau} I_{1-\frac{1}{k}}(r, j+1).$$

The difference $A - B$ results in Equation (6).

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