# BAYESIAN FREQUENCY ANALYSIS OF EXTREME RIVER DISCHARGES

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Probabilistic design of river dikes is usually based on estimates of a design discharge. In The Netherlands, a design discharge is defined as the discharge with an average return period of 1,250 years. Design discharges are currently estimated using classical statistical methods. A shortcoming of this approach is that statistical uncertainties are not taken into account and that probability distributions are given equal weight. In the paper, a method based on Bayesian statistics is presented. Using Bayes' theorem, the prior distribution representing information about the uncertainty of the statistical parameters can be updated to the posterior distribution as soon as data becomes available. Seven predictive probability distributions are investigated for determining extreme quantiles of discharges: the exponential, Rayleigh, normal, lognormal, gamma, Weibull and Gumbel. The Bayesian method has been successfully applied to estimate the design discharge of the river Rhine while taking account of the statistical uncertainties involved. The Bayes estimates are compared to the classical maximum-likelihood estimates. Furthermore, so-called Bayes factors are used to determine weights corresponding to how well a probability distribution fits the observed data; that is, the better the fit, the higher the weighting.

*Keywords:* Bayesian analysis, non-informative prior distribution, posterior distribution, maximum likelihood, river discharges, Bayes weights.

## 1 Introduction

Probabilistic design of river dikes is usually based on estimates of the design discharge. In The Netherlands, the design discharge is defined as an extreme discharge with an average return period of 1,250 years. Extreme quantiles, such as the design discharge are usually determined by fitting various probability distributions to the available observations. [See for example DH & EAC-RAND (1993), Castillo (1988), and Van Gelder (1999)]. Probability plots and goodness-of-fit tests (such as chi-square and Kolmogorov-Smirnov) are commonly used to select an appropriate distribution.

A major practical difficulty in fitting probability distributions is that there is often a limited amount of data for determining extreme quantiles. The associated return period is large compared with the length of the period of observation. In The Netherlands, observed flood discharges are available for a period of 98 years only. There is a large statistical uncertainty involved in estimating extreme quantiles when using these observations. The maximum-likelihood method has been recognised as one of the best parameter estimation methods (Galambos et al., 1994) and it is especially suitable when there is a large number

of observations. A drawback of the maximum-likelihood method is that statistical uncertainties cannot be taken into account.

Another consequence of sparse data is that more than one probability distribution seems to fit the observations and only a few can be rejected. These distributions usually lead to different extrapolated values and the goodness-of-fit tests for selecting the appropriate distribution are often inconclusive. The tests are more concentrated on the central part of the distribution than the tail. As an alternative, the Bayesian method can be used to determine weights for quantifying how well a probability distribution fits the observed data while taking account of the statistical uncertainties involved.

In this paper, a Bayesian method for estimating the design discharges is presented. Statistical uncertainties will be the subject of Section 2. Section 3 considers Bayesian estimation of both parameters and quantiles associated with large average return periods. Section 4 and 5 are devoted to determining non-informative prior distributions and Bayes weights, respectively. The annual maximum discharges of the river Rhine will be studied in Section 6. Section 7 ends with conclusions.

## 2 Statistical uncertainties

According to (amongst others) Slijkhuis et al. (1999) and Siu & Kelly (1998), uncertainties in risk analysis can primarily be divided into two categories: inherent uncertainties and epistemic uncertainties (see Figure 1). *Inherent uncertainties* represent randomness or variability in nature. For example, even in the event of sufficient data, one cannot predict the maximum discharge that will occur next year. The two main types of inherent uncertainty are inherent uncertainty in time (e.g., fluctuation of the discharge in time) and inherent uncertainty in space (e.g., fluctuation of a dike height in space). It is not possible to reduce inherent uncertainties completely. *Epistemic uncertainties* represent the lack of knowledge about a (physical) system. The two main types of epistemic uncertainty are statistical uncertainty (due to lack of sufficient data) and model uncertainty (due to lack of understanding the physics). Statistical uncertainty can be parameter uncertainty (when the parameters of the distribution are unknown) and distribution type uncertainty (when the type of distribution is unknown). In principle, epistemic uncertainties can be reduced as knowledge increases and more data becomes available.

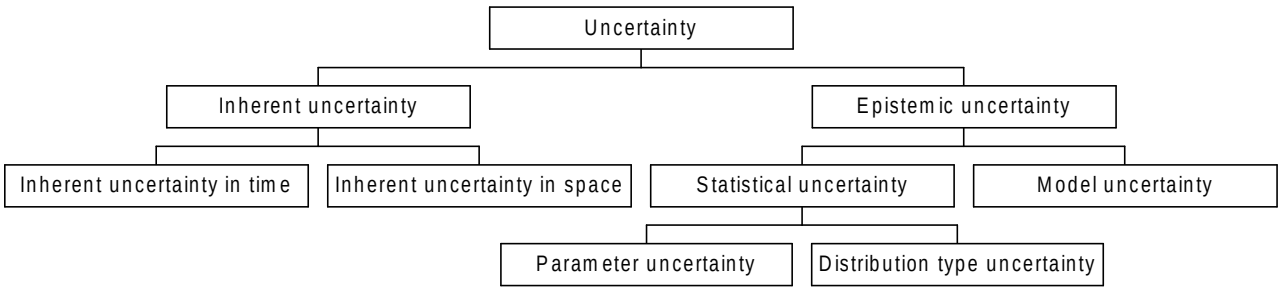


**Figure 1:** Types of uncertainty.

## 3 Bayesian estimation

The only statistical theory which combines modelling inherent uncertainty and statistical uncertainty is Bayesian statistics. The theorem of Bayes (1763) provides a solution to the problem of how to learn from data. In the framework of estimating the parameters $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_d)$ of a probability distribution $\ell(x|\boldsymbol{\theta})$, Bayes' theorem can be written as

$$\pi(\boldsymbol{\theta}|\mathbf{x}) = \frac{\ell(\mathbf{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})}{\int_{\boldsymbol{\theta}} \ell(\mathbf{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})d\boldsymbol{\theta}} = \frac{\ell(\mathbf{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})}{\pi(\mathbf{x})} \tag{1}$$

with

$\ell(\mathbf{x}|\boldsymbol{\theta})$ = the likelihood function of the observations $\mathbf{x} = (x_1,\ldots,x_n)$ when the parametric vector $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_d)$ is given,

$\pi(\boldsymbol{\theta})$ = the prior density of $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_d)$ before observing data $\mathbf{x} = (x_1,\ldots,x_n)$,

$\pi(\boldsymbol{\theta}|\mathbf{x})$ = the posterior density of $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_d)$ after observing data $\mathbf{x} = (x_1,\ldots,x_n)$,

and $\pi(\mathbf{x})$ = the marginal density of the observations $\mathbf{x} = (x_1,\ldots,x_n)$.

The likelihood function $\ell(x|\boldsymbol{\theta})$ represents the inherent uncertainty of a random variable $X$ when $\boldsymbol{\theta}$ is given, whereas the prior density $\pi(\boldsymbol{\theta})$ and the posterior density $\pi(\boldsymbol{\theta}|\mathbf{x})$ represent the statistical uncertainty in $\boldsymbol{\theta}$. This statistical uncertainty in $\boldsymbol{\theta}$ is parameter uncertainty. Using Bayes' theorem, we can update the prior distribution to the posterior distribution as soon as new observations become available. The more observations that are available, the smaller the parameter uncertainty. If a random variable $X$ has a probability density function $\ell(x|\boldsymbol{\theta})$ depending on the parametric vector $\boldsymbol{\theta}$, then the likelihood function $\ell(x_1,\ldots,x_n|\boldsymbol{\theta})$ of the independent observations $\mathbf{x} = (x_1,\ldots,x_n)$ is given by

$$\ell(\mathbf{x}|\boldsymbol{\theta}) = \ell(x_1,\ldots,x_n|\boldsymbol{\theta}) = \prod_{i=1}^{n} \ell(x_i|\boldsymbol{\theta}). \tag{2}$$

The marginal density $\pi(\mathbf{x})$ is obtained by integrating the likelihood $\ell(\mathbf{x}|\boldsymbol{\theta})$ over $\boldsymbol{\theta}$. Note that the maximum-likelihood estimate of the parametric vector $\boldsymbol{\theta}$ is defined as the estimate $\hat{\boldsymbol{\theta}}$, which maximises the likelihood function $\ell(x|\boldsymbol{\theta})$ as a function of $\boldsymbol{\theta}$.

The cumulative distribution function and the survival function of the random variable $X$ are denoted by $F(x|\boldsymbol{\theta})$ and $\overline{F}(x|\boldsymbol{\theta})$, respectively. The posterior predictive probability of exceeding $x_0$ is

$$\Pr\{X > x_0|\mathbf{x}\} = \int_{\boldsymbol{\theta}} \Pr\{X > x_0|\boldsymbol{\theta}\}\pi(\boldsymbol{\theta}|\mathbf{x})d\boldsymbol{\theta} = \int_{\boldsymbol{\theta}} \overline{F}(x_0|\boldsymbol{\theta})\pi(\boldsymbol{\theta}|\mathbf{x})d\boldsymbol{\theta}. \tag{3}$$

Besides representing parameter uncertainty on the basis of Bayesian statistics, distribution type uncertainty can also be taken into account using so-called Bayes factors or Bayes weights.

## 4 Non-informative priors

For the purpose of flood prevention, we would like the observations to 'speak for themselves', especially in comparison to the prior information. This means that the prior distribution should describe a certain 'lack of knowledge' or, in other words, should be as 'vague' as possible. For this purpose, so-called non-informative priors have been developed. A disadvantage of most non-informative priors is that these priors can be improper; that is, they often do not integrate to one. This disadvantage can be resolved by focussing on the posterior distributions rather than the prior distributions. As a matter of fact, formally carrying out the calculations of Bayes' theorem by combining an improper prior with observations often results in a proper posterior.

The pioneer in using non-informative priors was Bayes (1763) who considered a uniform prior. However, the use of uniform priors is criticised because of a lack of invariance under one-to-one transformations. As an example, let us consider an unknown parameter $\theta$ and suppose the problem has been parameterised in terms of $\phi = \exp\{\theta\}$. This is a one-to-one transformation, which should have no bearing on the ultimate result. The Jacobian of this transformation is given by $d\theta/d\phi = d\log\phi/d\phi = 1/\phi$. Hence, if the non-informative prior for $\theta$ is chosen to be uniform (constant), then the non-informative prior for $\phi$ should be proportional to $1/\phi$ to maintain consistency. Unfortunately, we cannot maintain consistency and choose both the non-informative priors for $\theta$ and $\phi$ to be constant.

The physicist Sir Jeffreys (1961, Chapters 3-4) was the first to produce an alternative to solely using uniform non-informative priors. His main motivation for deriving non-informative priors (currently known as Jeffreys priors) were invariance requirements for one-to-one transformations. In a multi-parameter setting, Jeffreys prior takes account of dependence between the parameters. For decades, there has been a discussion going on whether the multivariate Jeffreys rule is appropriate. We believe that the following statement made by Dawid (1999) is right: "we do not consider it as generally appropriate to use other improper priors than the Jeffreys measure for purposes of 'fully objective' formal model comparison". The main advantage of the Jeffreys prior is that it is always both invariant under transformations and dimensionless.

As an example, the multivariate Jeffreys prior for the normal model with unknown mean $\mu$ and unknown standard deviation $\sigma$ is

$$J(\mu,\sigma) = \frac{\sqrt{2}}{\sigma^2} d\mu d\sigma .$$

It can be easily seen that the above prior is dimensionless: i.e., $d\mu$, $d\sigma$, and $\sigma$ have the same dimension. For other examples, see the Appendix. Because the non-dimensionality argument is rather sound (from a physics point of view), we propose to use the multivariate Jeffreys measure for the purpose of model comparison.

In explaining the derivation of non-informative Jeffreys priors, we refer to Box & Tiao (1973, Section 1.3). Let $\mathbf{x} = (x_1,\ldots,x_n)$ be a random sample from a multi-parameter probability distribution with likelihood function $\ell(x|\theta)$. When the probability distribution obeys certain regularity conditions, then for sufficiently large $n$, the posterior density function of the parametric vector $\theta$ is approximately normal, and remains approximately normal under mild one-to-one transformations of $\theta$. As a consequence, the prior

distribution for $\theta$ is approximately non-informative if it is taken proportional to the square root of Fisher's information measure. In mathematical terms, the elements of this matrix are

$$I_{ij}(\theta) = E\left( -\frac{\partial^2 \log \ell(X|\theta)}{\partial\theta_i \partial\theta_j} \right), \quad i,j = 1,\ldots,d,$$

and the corresponding non-informative Jeffreys prior is defined by

$$J(\theta) \propto \sqrt{I(\theta)} = \sqrt{\det I_{ij}(\theta)}, \quad i,j = 1,\ldots,d.$$

## 5 Bayes factors and Bayes weights

The Bayesian approach to hypothesis testing originates from the work of Jeffreys (1961). He developed a methodology for quantifying the evidence in favour of a scientific theory using the so-called Bayes factor. This factor is the posterior odds of the null hypothesis when the prior probability on the null is one-half. A recent overview on Bayes factors can be found in Kass & Raftery (1995).

Assume the data $\mathbf{x} = (x_1,\ldots,x_n)$ to have arisen under one of $m$ models $H_k$, $k = 1,\ldots,m$. These hypotheses represent $m$ marginal probability densities $\pi(\mathbf{x}|H_k)$, $k = 1,\ldots,m$. Given the prior probabilities $p(H_k)$, $k = 1,\ldots,m$, the data produce the posterior probabilities $p(H_k|\mathbf{x})$, $k = 1,\ldots,m$, where $\sum_{j=1}^m p(H_j) = 1$ and $\sum_{j=1}^m p(H_j|\mathbf{x}) = 1$. These posterior probabilities can be obtained using Bayes' theorem as follows:

$$p(H_k|\mathbf{x}) = \frac{\pi(\mathbf{x}|H_k)p(H_k)}{\sum_{j=1}^m \pi(\mathbf{x}|H_j)p(H_j)}, \quad k = 1,\ldots,m, \tag{4}$$

where

$$B_{jk} = \frac{\pi(\mathbf{x}|H_j)}{\pi(\mathbf{x}|H_k)}, \quad j,k = 1,\ldots,m,$$

is denoted by the Bayes factor. The marginal densities of the data under $H_k$, $\pi(\mathbf{x}|H_k)$, can be obtained by integrating with respect to the probability distribution of the uncertain parametric vector $\theta_k = (\theta_{1k},\ldots,\theta_{dk})$ with number of parameters $d$:

$$\pi(\mathbf{x}|H_k) = \int \ell(\mathbf{x}|\theta_k, H_k)\pi(\theta_k|H_k)\, d\theta_k, \tag{5}$$

where $\pi(\theta_k|H_k)$ is the prior density of $\theta_k$ under $H_k$ and $\ell(\mathbf{x}|\theta_k, H_k)$ is the likelihood function of the data $\mathbf{x}$ given $\theta_k$.

A difficulty in using non-informative improper priors for calculating Bayes factors is that the prior odds, and thus the Bayes factor, may be undefined. The reason for this is that strictly speaking, the prior probability $p(H_k)$ is defined as

$$p(H_k) = w(H_k)\int J(\theta_k|H_k)\, d\theta_k,$$

where the integral over the non-informative Jeffreys prior $J(\boldsymbol{\theta}_k|H_k)$ is often infinite and $w(H_k)$ is the prior weight. However, according to Dawid (1999), this problem can be resolved by redefining the posterior odds as

$$\frac{p(H_j|\mathbf{x})}{p(H_k|\mathbf{x})} = \frac{\pi(\mathbf{x}|H_j)}{\pi(\mathbf{x}|H_k)} \times \frac{w(H_j)}{w(H_k)}, \quad j,k = 1,\ldots,m. \tag{6}$$

This posterior odds is well-defined so long as both integrals in it converge, which will typically be the case so long as the sample size $n$ is large enough. For the seven probability distributions considered in this paper (see the Appendix), the marginal densities of the data do indeed converge. Using Eqs. (4) and (6), the posterior probability of model $H_k$ being correct can now rewritten as

$$p(H_k|\mathbf{x}) = \frac{\pi(\mathbf{x}|H_k)w(H_k)}{\sum_{j=1}^{m}\pi(\mathbf{x}|H_j)w(H_j)}, \quad k = 1,\ldots,m. \tag{7}$$

It remains to choose the prior weights $w(H_k)$. For formal model comparison, we propose to use equal prior weights: i.e., $w(H_k) = 1/m,\ k = 1,\ldots,m$.

The posterior predictive probabilities of exceeding $x_0$ are calculated using the non-informative Jeffreys prior. Using the Bayes weights $p(H_k|\mathbf{x}),\ k = 1,\ldots,m$, the weighted predictive probability of exceeding $x_0$ is then defined by

$$\Pr\{X > x_0|\mathbf{x}\} = \sum_{k=1}^{m} p(H_k|\mathbf{x}) \Pr\{X > x_0|H_k,\mathbf{x}\}, \tag{8}$$

where $\Pr\{X > x_0|H_k,\mathbf{x}\}$ is the predictive probability of exceeding $x_0$ under likelihood model $H_k,\ k = 1,\ldots,m$.

Both the marginal densities and the predictive exceedance probabilities have been obtained by combining analytic calculation (if possible) and numerical integration (iterative quadrature).

## 6 Design discharge of the river Rhine

A statistical analysis based on a modified maximum-likelihood method applied to annual maximum discharges of the river Rhine at Lobith resulted in three distributions which could not be rejected. These three distributions are the lognormal, gamma and Gumbel. Because none of these distributions could be clearly identified as being the best, DH & EAC-RAND (1993) gave these three distributions equal weight. The design discharge is then defined as the average of the three corresponding design discharges. This method has been used to determine the current Dutch design discharges.

Recently, Van Gelder (1999) proposed to fit various probability distributions to the data and to attach different weights to these distributions according to how good the fits are. This can be done either by using Bayes weights (Van Gelder, 1999) or linear regression weights (Tang, 1970). Van Gelder et al. (1999) showed that Bayes weights perform better than linear regression weights.

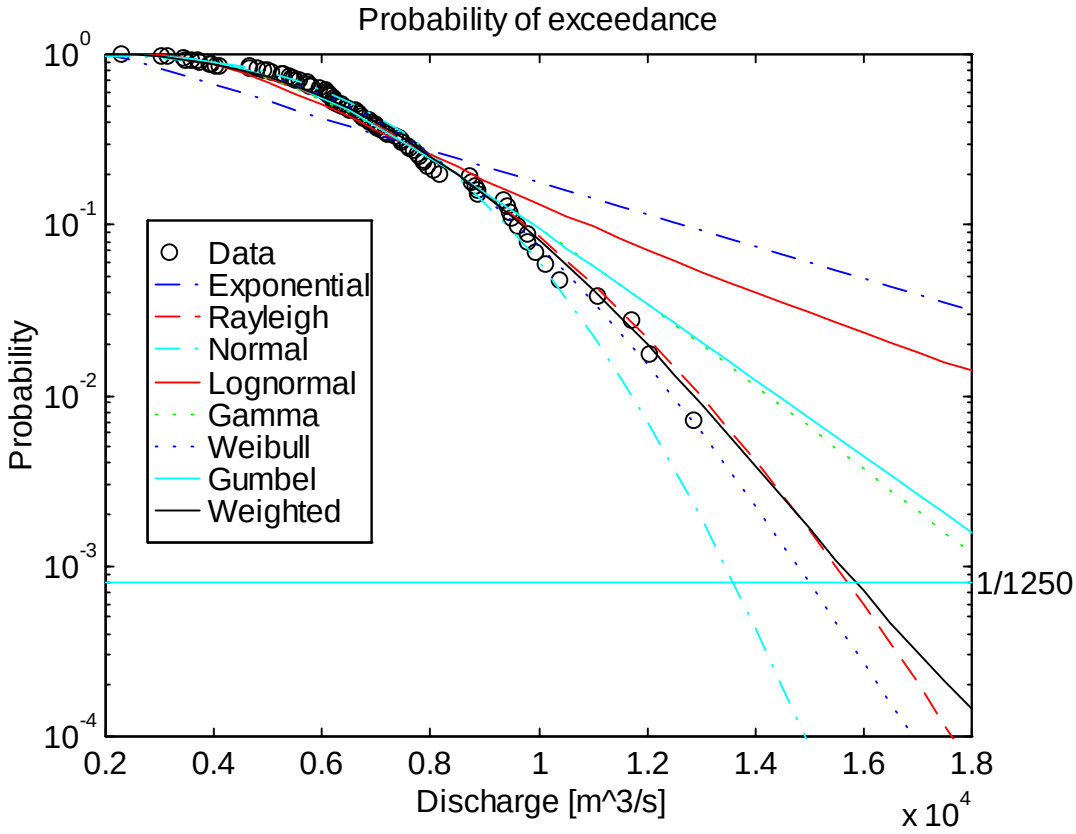


**Figure 2:** Predictive exceedance probability of annual maximum river Rhine discharge.

Bayesian analysis has been applied to the annual maximum discharges of the river Rhine at Lobith during the period 1901-1998. The Bayes weights (7) have been determined for seven probability distributions: the exponential, Rayleigh, normal, lognormal, gamma, Weibull and Gumbel. The Bayes weights largely depend on the location parameter. For proper model selection, we therefore propose to use the same location parameter for all seven distributions. On the basis of a statistical analysis, the location parameter can best be chosen to be 2,125 $m^3/s$. This location parameter follows by maximising the weighted marginal density of the observations, where Bayes weights have been attached to the seven individual marginal densities. The method of determining the location parameter, as well as a sensitivity analysis, will be presented in a forthcoming paper. For location parameter 2,125 $m^3/s$, the Bayes weights of the seven probability distributions can be found in Table 1. The Rayleigh and Weibull distribution appear to fit best with Bayes weights of 57% and 32%, respectively. The Bayes estimate of the design discharge with an average return period of 1,250 years is 15,860 $m^3/s$. Figure 2 shows both the empirical exceedance probability and the predictive exceedance probabilities. Using the maximum-likelihood method combined with the Bayes weights, the estimate of the design discharge decreases to 15,550 $m^3/s$. As expected, taking account of parameter uncertainty results in larger design discharges. The Bayesian approach seems to be promising for determining future design discharges.

| Bayes weight | Exponential | Rayleigh | Normal | Lognormal | Gamma | Weibull | Gumbel |
|---|---|---|---|---|---|---|---|
| Prior | 0.1429 | 0.1429 | 0.1429 | 0.1429 | 0.1429 | 0.1429 | 0.1429 |
| Posterior | 0.0000 | 0.5704 | 0.0480 | 0.0000 | 0.0097 | 0.3165 | 0.0554 |

**Table 1:** Prior and posterior Bayes weights for the annual maximum river Rhine discharge.

## 7 Conclusions

In this paper, the discharge of the Rhine at Lobith with an average return period of 1,250 years has been determined taking account of the statistical uncertainties involved. Statistical uncertainty occurs due to a lack of data. It can be subdivided into parameter uncertainty (when the parameters of a distribution are unknown) and distribution type uncertainty (when the type of distribution is unknown). Bayes estimates and Bayes weights can be used to account for parameter uncertainty and distributon type uncertainty, respectively. Using Bayes weights, it is possible to discriminate between different probability models and to quantify how well a distribution fits the data. The design discharge increases when taking the statistical uncertainties properly into account.

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**Appendix: Probability distributions and their Jeffreys priors**

This Appendix contains the probability distributions which are considered in the statistical analysis of the annual maximum discharges, as well as their non-informative Jeffreys priors. Special care has been given to deriving possible constants in the Jeffreys prior. The reason for this is that we agree with Dawid (1999), who stated that, "for the purposes of 'objective' model comparison, there is nothing to be gained by rescaling (...), and that the actual Jeffreys measure should be used".

*Exponential distribution*
A random variable $X$ has an exponential distribution with scale parameter $\theta > 0$ if the probability density function of $X$ is given by

$$\mathrm{Ex}(x|\theta) = \frac{1}{\theta}\exp\left\{-\frac{x}{\theta}\right\} I_{(0,\infty)}(x),$$

where $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$ for every set $A$. The Jeffreys prior is

$$J(\theta) = \frac{1}{\theta}.$$

*Rayleigh distribution*
A random variable $X$ has a Rayleigh distribution with quasi-scale parameter $\theta > 0$ if the probability density function of $X$ is given by

$$\mathrm{Ra}(x|\theta) = \frac{2x}{\theta}\exp\left\{-\frac{x^2}{\theta}\right\} I_{(0,\infty)}(x).$$

The Jeffreys prior for the Rayleigh distribution is

$$J(\theta) = \frac{1}{\theta}.$$

*Normal distribution*
A random variable $X$ has a normal distribution with mean $m$ and precision $r > 0$ if the probability density function of $X$ is given by

$$\mathrm{N}(x|m,r) = \left(\frac{r}{2\pi}\right)^{\frac{1}{2}} \exp\left\{-\frac{r}{2}(x-m)^2\right\}.$$

The joint Jeffreys prior of the mean $m$ and precision $r$ of a normal distribution is

$$J(m,r) = \frac{1}{\sqrt{2r}}.$$

*Lognormal distribution*
A random variable $X$ has a lognormal distribution with parameters $m$ and $r > 0$ if the probability density function of $X$ is given by

$$\mathrm{LN}(x|m,r) = \left(\frac{r}{2\pi}\right)^{\frac{1}{2}} \frac{1}{x}\exp\left\{-\frac{r}{2}(\log(x)-m)^2\right\} I_{(0,\infty)}(x).$$

Hence, if $\log(X)$ has a normal distribution, then $X$ has a lognormal distribution. The joint Jeffreys prior of the parameters $\mu$ and $r$ of a lognormal distribution is

$$J(m,r) = \frac{1}{\sqrt{2r}}.$$

*Gamma distribution*
A random variable $X$ has a gamma distribution with shape parameter $a > 0$ and scale parameter $b > 0$ if the probability density function of $X$ is given by

$$\mathrm{Ga}(x|a,b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp\{-bx\} I_{(0,\infty)}(x),$$

where $\Gamma(a) = \int_{t=0}^{\infty} t^{a-1} e^{-t} dt$ is the gamma function for $a > 0$. The (approximate) Jeffreys prior for the gamma distribution is

$$J(a,b) = \frac{\sqrt{a\psi'(a) - 1}}{b} \approx \frac{1}{b\sqrt{a}},$$

where the function $\psi'(a)$ is the first derivative of the digamma function:

$$\psi'(a) = \frac{\partial \psi(a)}{\partial a} = \frac{\partial}{\partial a}\left[\frac{\Gamma'(a)}{\Gamma(a)}\right]$$

for $a > 0$. Because calculating the first derivative of the digamma function is time consuming, Hora & Iman (1990) propose to approximate $a\psi'(a) - 1$ by $1/a$.

*Weibull distribution*
A random variable $X$ has a Weibull distribution with shape parameter $a > 0$ and quasi-scale parameter $b > 0$ if the probability density function of $X$ is given by

$$\mathrm{We}(x|a,b) = \frac{a}{b} x^{a-1} \exp\left\{-\frac{x^a}{b}\right\} I_{(0,\infty)}(x).$$

The Jeffreys prior for the Weibull distribution is

$$J(a,b) = \frac{1}{ab}\frac{\pi}{\sqrt{6}}.$$

*Gumbel distribution*
A random variable $X$ has a Gumbel distribution with location parameter $a$ and scale parameter $b > 0$ if the probability density function of $X$ is given by

$$\mathrm{Gu}(x|a,b) = \frac{1}{b}\exp\left\{-\frac{x-a}{b}\right\}\exp\left\{-\exp\left\{-\frac{x-a}{b}\right\}\right\}.$$

The Jeffreys prior for the Gumbel distribution is

$$J(a,b) = \frac{1}{b^2}\frac{\pi}{\sqrt{6}}.$$